SEC 3.5 INVERSES OF MATRICES

Remember: to bring matrix in REF we applied row operations, e.g.

$$\underset{\|}{\underline{\underline{A}}}\quad\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 1 & 5 & 6 & 8 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 3 & 3 & 4 \end{array}\right]\quad\underset{\|}{\underline{\underline{B}}}$$

We can write this operation also as

$$\boxed{\underline{\underline{A}}} \xrightarrow{\begin{array}{l} R_1 \to 1\cdot R_1 + 0 R_2 + 0\cdot R_3 \\ R_2 \to 0\cdot R_1 + 1\cdot R_2 + 0\cdot R_3 \\ R_3 \to -1\cdot R_1 + 0\cdot R_2 + 1\cdot R_3 \end{array}} \boxed{\underline{\underline{B}}}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \underline{\underline{C}}$$

The coefficients form a matrix themselves and more over

$$\underline{\underline{C}} \cdot \underline{\underline{A}} = \underline{\underline{B}}.$$

Every row operation we've performed is equivalent to a left multiplication by an "invertible" $n \times n$ matrix.

The process of elimination of variables from the viewpoint of matrices is the following.

(1) You write your system in matrix form as follows

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + \dots + a_{2n}x_n = b_2 \\ \quad\vdots \qquad\qquad\qquad\quad \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_n \end{cases} \longleftrightarrow \underline{\underline{A}} \cdot \underline{x} = \underline{b}$$

where $[\underline{\underline{A}}]_{ij} = a_{ij}$

(2) Multiply both sides of the equation with well-chosen matrices $\underline{\underline{C_1}}, \underline{\underline{C_2}}, \underline{\underline{C_3}}, \dots, \underline{\underline{C_k}}$ that perform the row operations to bring the LHS in RREF. Let $\underline{\underline{C}} = \underline{\underline{C_k}} \cdot \underline{\underline{C_{k-1}}} \cdot \dots \cdot \underline{\underline{C_2}} \cdot \underline{\underline{C_1}}$ then the new system becomes

$$(\underline{\underline{C}} \cdot \underline{\underline{A}})\,\underline{x} = \underline{\underline{C}} \cdot \underline{b}$$

(3) Once $(\underline{\underline{C}} \cdot \underline{\underline{A}})$ is in RREF we convert back to a system and set up the solutions.

Special Scenario

If $\underline{\underline{A}}$ is an $n \times n$ matrix and there exists a sequence of row operations to transform the system

$$\left[\begin{array}{c|c} \underline{\underline{A}} & \underline{b} \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & \dots & 0 & \\ 0 & 1 & & \vdots & \\ \vdots & & \ddots & 0 & \tilde{\underline{b}} \\ 0 & \dots & \dots & 1 & \end{array}\right]$$

unit matrix $\underline{\underline{1}}_{n \times n}$ $\qquad \tilde{\underline{b}} = \underline{\underline{C}} \cdot \underline{b} = \underline{\underline{A}}^{-1} \cdot \underline{b}$

then $\underline{\underline{A}}$ is an invertible matrix and the matrix $\underline{\underline{C}} = \underline{\underline{C_k}} \cdot \underline{\underline{C_{k-1}}} \cdot \dots \cdot \underline{\underline{C_2}}\,\underline{\underline{C_1}}$ used to bring the system into RREF is the inverse of $\underline{\underline{A}}$; $\underline{\underline{A}}^{-1}$.

With other words, if the system of equations can be written in matrix form like

$$\underline{\underline{A}}\,\underline{x} = \underline{b}$$

where $\underline{\underline{A}}$ is an $n \times n$ invertible matrix then we can solve the system by multiplying on the left by $\underline{\underline{A}}^{-1}$ to get

$$\underline{\underline{A}}^{-1}\underline{\underline{A}}\,\underline{x} = \underline{\underline{A}}^{-1}\underline{b}$$

$$\Leftrightarrow \quad \underline{\underline{\mathbb{1}}}_{n \times n}\,\underline{x} = \underline{\underline{A}}^{-1}\underline{b}$$

$$\underline{x} = \underline{\underline{A}}^{-1}\,\underline{b}$$

EXAMPLE: Given that $\begin{bmatrix} 9 & -4 \\ -2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$ solve $\begin{cases} 9x - 4y = 2 \\ -2x + y = -1 \end{cases}$

Q: how to find $\underline{\underline{A}}^{-1}$? (if it exists)

A: When we reduce $\underline{\underline{A}}$ to RREF we can keep track of the matrix used to transform $\underline{\underline{A}}$ to $\underline{\underline{\mathbb{1}}}_{n \times n}$ as follows. Rather than row reducing $\underline{\underline{A}}$ we row reduce the matrix

$$\left[\begin{array}{c|c} \underline{\underline{A}} & \underline{\underline{\mathbb{1}}}_{n \times n} \end{array}\right]$$

Row reducing this extended matrix is the same as multiplying both matrices simultaneously by the same matrices $\underline{\underline{C}}_1, \underline{\underline{C}}_2, \ldots, \underline{\underline{C}}_k$, so we get a new matrix that looks like

$$\underline{\underline{C}}\,\underline{\underline{A}} = \underline{\underline{\mathbb{1}}}_{n \times n} \quad \text{so} \quad \underline{\underline{C}} = \underline{\underline{A}}^{-1}$$

$$\left[\begin{array}{c|c} \underline{\underline{C}}\,\underline{\underline{A}} & \underline{\underline{C}}\,\underline{\underline{\mathbb{1}}}_{n \times n} \end{array}\right] = \left[\begin{array}{c|c} \underline{\underline{\mathbb{1}}}_{n \times n} & \underline{\underline{C}} \end{array}\right]$$

EXAMPLES (1) Solve the system of lin eqns

$$\begin{cases} 5x + 4y = 1 \\ 6x + 5y = 2 \end{cases}$$

by (a) rewriting it in the form $\underline{\underline{A}}\,\underline{x} = \underline{b}$, $\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, (b) finding $\underline{\underline{A}}^{-1}$; and (c) finding $\underline{x}$ via $\underline{x} = \underline{\underline{A}}^{-1}\underline{b}$

(a) $\begin{bmatrix} 5 & 4 \\ 6 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(b) to find $\underline{\underline{A}}^{-1}$, lets bring $\left[\begin{array}{cc|cc} 5 & 4 & 1 & 0 \\ 6 & 5 & 0 & 1 \end{array}\right]$ in RREF.

$$\begin{bmatrix} 5 & 4 & 1 & 0 \\ 6 & 5 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 \cdot 5]{R_1 \cdot 6} \begin{bmatrix} 30 & 24 & 6 & 0 \\ 30 & 25 & 0 & 5 \end{bmatrix}$$

$$\xrightarrow{R_2 - R_1} \begin{bmatrix} 30 & 24 & 6 & 0 \\ 0 & 1 & -6 & 5 \end{bmatrix} \xrightarrow{R_1 - 24 \cdot R_2} \begin{bmatrix} 30 & 0 & 25.6 & -24.5 \\ 0 & 1 & -6 & 5 \end{bmatrix}$$

$$\xrightarrow{R_1/30} \left[\begin{array}{cc|cc} 1 & 0 & 5 & -4 \\ 0 & 1 & -6 & 5 \end{array}\right] \quad \text{so} \quad \underline{\underline{A}}^{-1} = \begin{bmatrix} 5 & -4 \\ -6 & 5 \end{bmatrix}$$

(c) $\begin{bmatrix} x \\ y \end{bmatrix} = \underline{x} = \underline{\underline{A}}^{-1}\,\underline{b} = \begin{bmatrix} 5 & -4 \\ -6 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

(2) Find the inverse of $\underline{\underline{A}} = \begin{bmatrix} -2 & 3 & 1 \\ 1 & -1 & -2 \\ -1 & 3 & -7 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} -2 & 3 & 1 & 1 & 0 & 0 \\ 1 & -1 & -2 & 0 & 1 & 0 \\ -1 & 3 & -7 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -1 & -2 & 0 & 1 & 0 \\ -2 & 3 & 1 & 1 & 0 & 0 \\ -1 & 3 & -7 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\substack{R_2 + 2R_1 \\ R_3 + R_1}} \left[\begin{array}{cccccc} 1 & -1 & -2 & 0 & 1 & 0 \\ 0 & 1 & -3 & 1 & 2 & 0 \\ 0 & 2 & -9 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_2} \left[\begin{array}{cccccc} 1 & -1 & -2 & 0 & 1 & 0 \\ 0 & 1 & -3 & 1 & 2 & 0 \\ 0 & 0 & -3 & -2 & -3 & 1 \end{array}\right] \xrightarrow{R_3\left(\frac{-1}{3}\right)} \left[\begin{array}{cccccc} 1 & -1 & -2 & 0 & 1 & 0 \\ 0 & 1 & -3 & 1 & 2 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 1 & -\frac{1}{3} \end{array}\right]$$

$$\xrightarrow{\substack{R_1 + 2R_3 \\ R_2 + 3R_3}} \left[\begin{array}{cccccc} 1 & -1 & 0 & \frac{4}{3} & 3 & -\frac{2}{3} \\ 0 & 1 & 0 & 3 & 5 & -1 \\ 0 & 0 & 1 & \frac{2}{3} & 1 & -\frac{1}{3} \end{array}\right]$$

$$\xrightarrow{R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{13}{3} & 8 & -\frac{5}{3} \\ 0 & 1 & 0 & 3 & 5 & -1 \\ 0 & 0 & 1 & \frac{2}{3} & 1 & -\frac{1}{3} \end{array}\right]$$

So $$\underline{\underline{A}}^{-1} = \left[\begin{array}{ccc} \frac{13}{3} & 8 & -\frac{5}{3} \\ 3 & 5 & -1 \\ \frac{2}{3} & 1 & -\frac{1}{3} \end{array}\right]$$

SOLVING MULTIPLE SYSTEMS AT ONCE

If you have multiple systems of lin equations whose coefficient matrix is the same matrix $\underline{\underline{A}}$, so

$$\underline{\underline{A}}\,\underline{x}_1 = \underline{b}_1, \quad \underline{\underline{A}}\,\underline{x}_2 = \underline{b}_2, \ldots, \underline{\underline{A}}\,\underline{x}_k = \underline{b}_k$$

then you can write this system as 1 matrix equation as follows:

$$\underline{\underline{A}} \cdot \underbrace{\left[\begin{array}{c:c:c:c} \underline{x}_1 & \underline{x}_2 & \cdots & \underline{x}_k \end{array}\right]}_{\underline{\underline{X}}} = \underbrace{\left[\begin{array}{c:c:c:c} \underline{b}_1 & \underline{b}_2 & \cdots & \underline{b}_k \end{array}\right]}_{\underline{\underline{B}}}$$

or $$\underline{\underline{A}} \cdot \underline{\underline{x}} = \underline{\underline{B}}$$

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