SEC 3.3. REDUCED ROW ECHELON MATRICES

Recap

DEFINITIONS A matrix is in

- row echelon form (REF) if
  (1) All rows containing only zeros lie below all rows containing a nonzero element.
  (2) For each row, its leading coeff lies to the right of the leading coeffs of the rows above
- reduced row echelon form (RREF) if it is in REF and
  (1) Each leading coeff is 1
  (2) Each leading coeff is the only nonzero coeff in its column

THEOREM A RREF of a matrix is unique.

EXAMPLE Let $\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & \alpha \end{bmatrix}$

QUESTION (1) Reduce $\underline{\underline{A}}$ to RREF, (2) for what value(s) of α is the system corresponding to $\underline{\underline{A}}$ consistent?

$$\xrightarrow[R_3 - 9R_1]{R_2 - 5R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & \alpha-36 \end{bmatrix} \xrightarrow[R_3\cdot(-1)]{R_2\cdot\left(\frac{-1}{4}\right)} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 8 & 16 & 36-\alpha \end{bmatrix}$$

$$\xrightarrow{R_3 - 8R_2} \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 12-\alpha \end{array}\right] \xrightarrow{R_1 - 2R_2} \left[\begin{array}{cccc} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 12-\alpha \end{array}\right]$$

System is only consistent when $\alpha = 12$

$$\begin{cases} x \quad + z = -2 \\ y + 2z = 3 \\ 0 = 12 - \alpha \end{cases}$$

SOME SPECIAL SYSTEMS

DEFINITION A system of linear equations is called homogenous if it's of the form

$$\begin{cases} \cdots & = 0 \\ & = 0 \\ & = 0 \\ & \vdots \\ & = 0 \end{cases} \longleftrightarrow \left[\begin{array}{c|c} & 0 \\ \cdots & 0 \\ & \vdots \\ & 0 \end{array}\right]$$

EXAMPLES (1) $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 3 & 9 & 9 & 0 \end{array}\right]$ (2) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

THEOREM If a homogenous equation has more variables (or unknowns) than equations, it has an infinite number of solutions.
The number of parameters appearing the solution equals the number of variables − number of eqns.

SPECIAL CASE OF INTEREST

THEOREM A homogenous system with as many equations as variables has a unique solution if and only if its RREF looks like

$$\left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{array}\right]$$

SEC 3.4. MATRIX OPERATIONS

Lets introduce some operations on matrices.

DEFINITIONS

- Two $m \times n$ matrices $\underline{\underline{A}}, \underline{\underline{B}}$ are equal if an only if $[\underline{\underline{A}}]_{ij} = [\underline{\underline{B}}]_{ij}$ for all i, j
- Let $\underline{\underline{A}}, \underline{\underline{B}}$ be two $m \times n$ matrices then $\underline{\underline{C}} = \underline{\underline{A}} + \underline{\underline{B}}$ has coeffs $[\underline{\underline{C}}]_{ij} = [\underline{\underline{A}}]_{ij} + [\underline{\underline{B}}]_{ij}$
- Let $\underline{\underline{A}}$ be a $m \times n$ matrix and c a real number then $\underline{\underline{B}} = c\underline{\underline{A}}$ has coeffs $[\underline{\underline{B}}]_{ij} = c[\underline{\underline{A}}]_{ij}$

DEFINITION A column vector is a $n \times 1$ matrix, a row vector is a $1 \times n$ matrix. Typically I use the following notation

$$\underline{x} = \boxed{\begin{matrix} \\ \\ \\ \end{matrix}} \quad \text{or} \quad y = \boxed{}$$

The multiplication of a $1 \times n$ row vector $\underline{a}$ with a $n \times 1$ column vector $\underline{b}$ is a number

$$c = [\underline{a}]_1 [\underline{b}]_1 + [\underline{a}]_2 [\underline{b}]_2 + \dots + [\underline{a}]_n [\underline{b}]_n$$

DEFINITION • The i th row of a matrix $\underline{\underline{A}}$ is also denoted by $[\underline{\underline{A}}]_{i:}$, the j th column of $\underline{\underline{A}}$ is denoted by $[\underline{\underline{A}}]_{:j}$

• Let $\underline{\underline{A}}$ be an $m \times k$ matrix and $\underline{\underline{B}}$ be a $k \times n$ matrix. The matrix $\underline{\underline{C}}$ has elements

$$[\underline{\underline{C}}]_{ij} = [\underline{\underline{A}}]_{i:} [\underline{\underline{B}}]_{:j}$$

EXAMPLE (1) Let $\underline{\underline{A}} = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 3 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$

$$\underline{\underline{B}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix}$$

$$\underline{\underline{A}} \cdot \underline{\underline{B}} = \begin{bmatrix} R_1 \cdot C_1 & R_1 \cdot C_2 & R_1 \cdot C_3 \\ R_2 \cdot C_1 & R_2 \cdot C_2 & R_2 \cdot C_3 \end{bmatrix}$$

$R_1 \cdot C_1 = 1 \cdot 1 - 1 \cdot 0 + 0 \cdot 1 + 2 \cdot 0 = 1$

$R_1 \, C_2 = 1 \cdot 0 - 1 \cdot 2 + 0 \cdot 0 + 2 \cdot 0 = -2$

$R_1 \, C_3 = 1 \cdot 1 - 1 \cdot 0 + 0 \cdot 1 + 2 \cdot 1 = 3$

$R_2 \cdot C_1 = 3 \cdot 1 + 0 \cdot 0 + 1 \cdot 1 + 1 \cdot 0 = 4$

$R_2 \, C_2 = 3 \cdot 0 + 0 \cdot 2 + 1 \cdot 0 + 1 \cdot 0 = 0$

$R_2 \cdot C_3 = 3 \cdot 1 + 0 \cdot 0 + 1 \cdot 1 + 1 \cdot 1 = 5$

$$\underline{\underline{A}} \cdot \underline{\underline{B}} = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 0 & 5 \end{bmatrix}$$

(2) $$\underline{\underline{A}} = \begin{array}{cc}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \hline a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33}\end{bmatrix} & \begin{array}{c} \underline{R_1} \\ \underline{R_2} \\ \underline{R_3} \end{array}\end{array} \qquad \underline{x} = \begin{array}{c} \underline{c_1} \\ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\end{array}, \quad \underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

then what does the following equation look like?

$$\underline{\underline{A}} \cdot \underline{x} = \underline{b}$$

$$\begin{bmatrix} \underline{R_1} \\ \hline \underline{R_2} \\ \hline \underline{R_3} \end{bmatrix} \begin{bmatrix} \underline{c_1} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\shortparallel$$

$$\begin{bmatrix} \underline{R_1} \cdot \underline{c_1} \\ \underline{R_2} \cdot \underline{c_1} \\ \underline{R_3} \cdot \underline{c_1} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\boxed{\begin{matrix} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 \\ a_{21} x_1 + a_{22} x_2 + a_{23} x_3 \\ a_{31} x_1 + a_{32} x_2 + a_{33} x_3 \end{matrix}} = \boxed{\begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix}}$$

PROPERTIES OF MATRIX MULTIPLICATION

(1) $\underline{\underline{A}}\,\underline{\underline{B}} \neq \underline{\underline{B}}\,\underline{\underline{A}}$

not always true. (it can be true for specific matrices however)

(2) Denote by $\mathbb{1}_{n\times n}$ the $n \times n$ matrix with 1's on the diagonal and 0's every where else, then for any $m \times n$ matrix $\underline{\underline{A}}$ and $n \times m$ matrix $\underline{\underline{B}}$

$$\underline{\underline{A}}\,\underline{\underline{\mathbb{1}}}_{n\times n} = \underline{\underline{A}}, \quad \underline{\underline{\mathbb{1}}}_{n\times n}\,\underline{\underline{B}} = \underline{\underline{B}}$$

(3) Let $\underline{\underline{A}}, \underline{\underline{B}}, \underline{\underline{C}}$ be matrices for which the following products are well defined

$$\underline{\underline{A}}\,(\underline{\underline{B}}\,\underline{\underline{C}}),\ (\underline{\underline{A}}\cdot\underline{\underline{B}})\,\underline{\underline{C}}$$

then $$\underline{\underline{A}}\,(\underline{\underline{B}}\,\underline{\underline{C}}) = (\underline{\underline{A}}\cdot\underline{\underline{B}})\,\underline{\underline{C}}$$

(4)
$$\underline{\underline{A}}\,(\underline{\underline{B}} + \underline{\underline{C}}) = \underline{\underline{A}}\,\underline{\underline{B}} + \underline{\underline{A}}\,\underline{\underline{C}}$$
$$(\underline{\underline{B}} + \underline{\underline{C}})\,\underline{\underline{A}} = \underline{\underline{B}}\,\underline{\underline{A}} + \underline{\underline{C}}\,\underline{\underline{A}}$$

(5) If $\underline{\underline{A}}\cdot\underline{\underline{B}} = \underline{\underline{0}}$ ← matrix with only zeros.
then $\underline{\underline{A}} = \underline{\underline{0}}$ or $\underline{\underline{B}} = \underline{\underline{0}}$?

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

(6) If $\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{A}}\,\underline{\underline{C}}$ then its possible that
$$\underline{\underline{B}} \neq \underline{\underline{C}}$$