SEC 3.2 MATRICES & GAUSSIAN ELIMINATION

In previous lecture we learned how to solve systems of lin eqns, e.g.

$$\begin{cases} \underset{\text{1st var}}{1x} + \underset{\text{2nd var}}{2y} + \underset{\text{3rd var}}{1z} = 4 \\ 3x + 8y + 7z = 20 \\ 2x + 7y + 9z = 23 \end{cases}.$$

If we always keep the variables in the same order in the equations then we can save a lot of work by not writing them at all. Likewise for the "+" and "=" signs. The following would be a more compact way of writing the above system

DEFINITIONS

$$\underline{\underline{A}} = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{array}\right] \leftarrow \underset{\text{rows}}{3} \times \underset{\text{columns}}{4} \text{ matrix}$$

double underline for matrices

The numbers are called its entries, elements, or coefficients.

this submatrix is called the coefficient matrix of the system of eqns.

Note conventions on brackets!

Operations on equations become operations on rows.

DEFINITION Two matrices are called row equivalent if they can be transformed into each other using a combination of the following 3 operations

(1) Swapping 2 rows

(2) Multiplying a row with a non zero number

(3) Adding a multiple of one row to another.

Theorem Two matrices are row equivalent if and only if they represent equivalent systems of equations.

EXAMPLE Solve
$$\begin{cases} 1x + 2y + 1z = 4 \\ 3x + 8y + 7z = 20 \\ 2x + 7y + 9z = 23 \end{cases}$$
by converting it to a matrix first.

In matrix form the system looks like

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{bmatrix} \xrightarrow[(R_3)-2(R_1)]{(R_2)-3(R_1)} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 1 & 7 & 15 \end{bmatrix}$$

$$\xrightarrow{\cdots} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

DEFINITION We say a matrix is in row echelon form if

(1) Any row of zeroes lies below any row that contains a non zero element.

(2) For any row, its first nonzero element lies to the right of the first nonzero elements of any of the rows above

EXAMPLES (1)

(2) $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 2 & 2 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, (3) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

COUNTEREXAMPLES

(1) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$, (2) $\begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$

Once a matrix is in $\overbrace{\text{row-echelon form}}^{\text{REF}}$ you could already solve the system via <u>back-substitution</u>

How does this work?

(1) Convert the matrix (in REF) back to a system

(2) For the last equation set all variables <u>except</u> the leading variable equal to different parameters.

(3) Solve the last equation for its leading variable

(4) Substitute the result in all other equations.

(5) Repeat steps (2)–(5) for that equation until all variables are known.

EXAMPLES

(1) for the system
$$\begin{cases} 1x + 2y + 1z = 4 \\ 3x + 8y + 7z = 20 \\ 2x + 7y + 9z = 23 \end{cases}$$
we found that its matrix is row equivalent to the following

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Lets apply the steps above to solve it.

(1) The system is
$$\begin{cases} x + 2y + z = 4 \\ \quad\;\; y + 2z = 4 \\ \qquad\quad\;\; z = 3 \end{cases}$$

(2) There is only the leading variable z so nothing to do.

(3) $z = 3$

(4) $\begin{cases} x + 2y + 3 = 4 \\ \quad\;\; y + 6 = 4 \end{cases}$

(2 For eqn $y + 6 = 4$) Nothing to do

(3) $y = -2$

(4) $\{ x - 4 + 3 = 4$ so $x = 5$

All together: $(x, y, z) = (5, -2, 3)$

(2) Solve
$$\begin{cases} x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 = 10 \\ 2x_1 - 4x_2 + 8x_3 + 3x_4 + 10x_5 = 7 \\ 3x_1 - 6x_2 + 10x_3 + 6x_4 + 5x_5 = 27 \end{cases}$$

First step: convert to matrix and bring in REF.

$$\begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{bmatrix} \xrightarrow[(R_3)-3(R_1)]{(R_2)-2(R_1)} \begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 2 & -1 & 8 & -13 \\ 0 & 0 & 1 & 0 & 2 & -3 \end{bmatrix}$$

$$\xrightarrow{R_2 \Leftrightarrow R_3} \begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 2 & -1 & 8 & -13 \end{bmatrix}$$

$$\xrightarrow{(R_3)-2(R_2)} \begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & -1 & 4 & -7 \end{bmatrix} \leftarrow \text{in REF}$$

Now we apply the back substitution to solve the system

(1)
$$\begin{cases} x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 = 10 \\ x_3 + 2x_5 = -3 \\ -x_4 + 4x_5 = -7 \end{cases}$$

(2) x_4 is leading. $x_5 = t$

(3) $x_4 = 7 - 4t$

(4) $\begin{cases} x_1 - 2x_2 + 3x_3 + 14 - 8t + t = 10 \\ x_3 + 2t = -3 \end{cases}$

(2 for $x_3 + 2t = -7$) Nothing to do

(3) $x_3 = -2t - 3$

(4) $\{ x_1 - 2x_2 - 6t - 9 + 14 - 8t + t = 10$

$\hookrightarrow \{ x_1 - 2x_2 = 13t - 5$

Set $x_2 = u$ then

$$x_1 = 13t + 2u - 5$$
$$x_2 = u$$
$$x_3 = -3 - 2t$$
$$x_4 = 7 - 4t$$
$$x_5 = t$$

SEC 3.3. REDUCED ROW ECHELON MATRICES

Because you can choose to swap rows at any time during Gaussian elimination the resulting REF of a matrix is not unique. There is a stricter version of a REF that is unique, however.

DEFINITION A matrix is in reduced row echelon form (RREF) if it is in REF and

(1) The leading coefficients of all rows are 1.

(2) The leading coefficients for all rows are the only nonzero element in their column.

EXAMPLES The matrices

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} = \underline{\underline{B}}$$

are in REF but not in RREF. It is easy to transform them to equivalent RREF matrices, though.

$$\underline{\underline{A}} : \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \xrightarrow[(R_3)\cdot\frac{1}{6}]{(R_2)\cdot\frac{1}{4}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{5}{4} \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 - \frac{5}{4}R_3]{R_1 - 3R_3} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\underline{B}} : \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow[R_3\cdot\frac{1}{3}]{R_2\cdot\frac{1}{2}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\cdots} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

A matrix with 1's on the diagonal and 0's everywhere else is called a UNIT MATRIX.

THEOREM A RREF of a matrix is unique.