CHAPTER 1

FIRST ORDER DIFFERENTIAL EQUATIONS

SEC 1.1. BASIC MODELS

DEFINITION A differential equation (DE) is an equation that relates one or more functions to their derivatives.

They play an important role in modeling phenomena in nature.

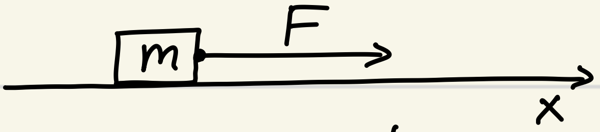
EXAMPLES (1) Population models: the speed at which a population grows is proportional to the size of the population.
↳ very naive...

Let P stand for the size of the population, then

$$\frac{dP}{d\underset{\substack{\uparrow \\ \text{time}}}{t}} = \underset{\substack{\uparrow \\ \text{growth factor} \\ \text{constant}}}{k}\,P$$

(2) Newton's second law

In 1 dimension



then

$$F = \frac{d\overset{\text{momentum}}{p}}{dt} = \frac{dmv}{dt}$$

(if m is constant)

$$= m\frac{dv}{dt}$$

$$= m\frac{d^2x}{dt^2}$$

Often we want to find x as a function of t. Note that F itself might depend on x and/or t.

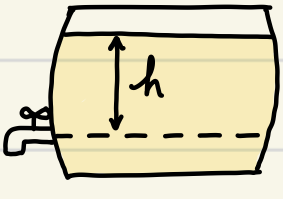
(3) Wave equation in 2D

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

where u is a function of 3 variables: $\overset{\text{time}}{t}, \overbrace{x, y}^{\text{position}}$ and equals the displacement of some quantity from its equilibrium.

(4) Student life.

The speed at which beer flows from a barrel depends on the height of the beer in the barrel. If we approximate the barrel as a cilinder, then



$$\frac{dh}{dt} = -k\sqrt{h}$$

(5) Non-example: population models with small populations. E.g., the number of cats in a neighboarhood is always an integer and as long as its population stays small a differential equation would not be suitable.

In this course we will restrict ourselves to certain DEs: Real ordinary DEs.

Real ↓ no complex numbers in equations

ordinary ↓ no partial derivatives

and we will only search for real solutions: functions that take real numbers and return real numbers.

continuation of sec 1.1.

The main goal of this course is to learn techniques for solving ODEs.
So far, in most courses, solving an equation meant finding one or more numerical values of some unknows like $x, y, \dots$ Solving an ODE, however, means finding all functions that satisfy the ODE. This means that, if e.g. $f(x)$ solves an ODE, then plugging $f(x)$ in that ODE results in an equation that is trivially true.

EXAMPLE (Population model with $k=1$)
The function $p(t) = e^t$ is a solution to the ODE

$$\circledast \qquad \frac{dP}{dt} = P$$

Indeed, if we plug $p = e^t$ in we get

$$\frac{de^t}{dt} = e^t \Rightarrow e^t = e^t \checkmark$$

Note that $p(t) = e^t$ is not the only solution to this ODE. You can verify that $p(t) = ce^t$ also solves the ODE for any constant c.

Actually most ODEs have an infinite number of solutions. Still, in reality there should only be 1 possible way in which a system evolves. To single out a solution, we need extra information. For, e.g., the population model ⊛, an initial value of the population at $t=0$ suffices.

EXAMPLE If the number of squirrels at Purdue evolves according to

$$\frac{dP}{dt} = P$$

and there are 100 squirrels at $t=0$, then what is the population of squirrels at any time t?
We know that $P(t) = ce^t$ solves the ODE and moreover that at $t=0$, $P=100$, or

$$P(0) = 100 = \left[ce^t\right]_{t=0} = ce^0 = c$$

So $c=100$ and thus

$$\boxed{P(t) = 100e^t}$$

NOTES • We assumed that all solutions to the ODE are of the form $P(t) = ce^t$. Soon we will see that this is, indeed, the case.

• Sometimes another initial condition is given. The value of c can still be found the same way: combine the given info with the general solution to obtain a particular one.

DEFINITION An ODE together with an initial condition is called an initial value problem.

CONVENTIONS

1) Typically we denote the unknown function in an ODE by $y(x)$ (or just y), $x(t)$ (or just x), or by a letter thats related to the quantity we're modeling, e.g., P for population.
2) Instead of $\frac{dy}{dx}$ we sometimes write y' or Dy. It should be clear from the context which symbol is the unknown function and which is the variable that function depends on. For $\frac{d^ny}{dx^n}$ we also write $y^{(n)}$ or D^ny.

DEFINITION The order of an ODE equals the highest derivative that appears in the ODE.

EXAMPLES

(1) $y' = y$: 1st order
(2) $(y')^2 = y$: 1st order
(3) $y''' + y = 0$: 3rd order
(4) $(y''+y')^2 - (y'')^2 - 2y''y' = 0$: 1st order

SEC 1.2. INTEGRALS AS GENERAL & PARTICULAR SOLUTIONS

If an ODE has the following form

$$y' = F(x) \longleftarrow \text{some known function of } x$$

we can integrate both sides to get

$$\int y'\,dx = \int F(x)\,dx$$
$$\Longleftrightarrow \quad y + c_y = \int F(x)\,dx$$
$$\Longleftrightarrow \quad y = \int F(x)\,dx - c_y$$

where c_y is an integration constant. Since it can have any value we might as well say that

$$y = \int F(x)\,dx + c$$

EXAMPLES

(1) $y' = \cos(x)$

$$\Rightarrow \quad y = \int \cos(x)\,dx + c_1$$
$$= \sin(x) + c_2 + c_1$$
$$\Rightarrow \quad \boxed{y = \sin(x) + c}$$

since $c_1 + c_2$ can take any value you can just combine them in one constant

From the example its clear that we can simplify the solution to $y' = F(x)$ to $y = \int F(x)\,dx$. Because of the integration constant there are an infinite number of solutions, one for each value of the integration constant.

A closely related ODE is the following:

$$\frac{d^n y}{dx^n} = F(x) \quad \text{with } n \text{ a positive integer}$$

The solution is obtained by integrating both sides of the equation n times:

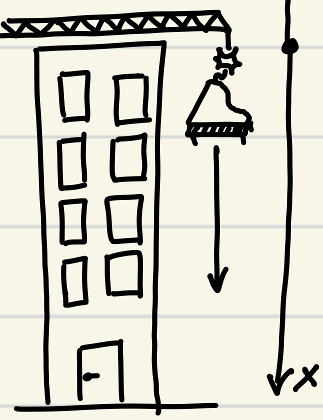
$$y = \underbrace{\int \cdots \int}_{n \text{ times}} F(x)\,dx \ldots dx \qquad \circledast$$

IMPORTANT NOTE Every time you compute an integral in $\circledast$ a new integration constant appears. When computing the next integral, you also need to integrate this (and all previous) constant. Eventually you should get something of the form

$$y = G(x) + C_1 \frac{x^{n-1}}{(n-1)!} + C_2 \frac{x^{n-2}}{(n-2)!} + \ldots + C_{n-1} x + C_n$$

where $n! = n\cdot(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

EXAMPLE Consider an object that is falling (in a vacuum) with a constant acceleration g.



By definition, the acceleration $a(t) = \frac{d^2x}{dt^2}$
Since $a(t) = g$ we have that

$$\frac{d^2x}{dt^2} = g$$

so

$$x(t) = \iint g \, dt \, dt$$

$$= \int (gt + c_1) dt$$

$$= \frac{g}{2}t^2 + c_1 t + c_2$$

In this case the constants have a clear physical interpretation.
If $t=0$ then $x = \frac{g}{2}\cdot 0 + c_1 \cdot 0 + c_2 = c_2$
so $c_2 = x(0)$, which is often written as x_0: the position at $t=0$
Likewise if $t=0$ then $v = \frac{dx}{dt} = \left[g \cdot t + c_1\right]_{t=0} = c_1$
so $c_1 = v(0) = v_0$: the velocity at $t=0$.
The general solution $\boxed{x(t) = \frac{g}{2}t^2 + v_0 t + x_0}$
depends on 2 integration constants: v_0 and x_0 which makes sense: to know the position of the object, we need to know its starting position and velocity.

NOTES

- You can also determine c_1, c_2 by giving 2 positions at two different times, 2 velocities at 2 different times or a velocity and a position at 2 (not necessarilly different) times. They only have a "nice" interpretation in terms of position and velocity at $t=0$, though.
- Can you see that the solution is of the form given in the important note?

- It is not possible to combine c_1, c_2 in one constant because of the variable t that appears in $c_1 t + c_2$. Not all functions $c_1 t + c_2$ can be written as ct or $t+c$, in contrast with all (constant) functions $c_1 + c_2$ or $c_1 \cdot c_2$ or $\frac{c_1}{c_2}$, etc that can be written as c.

- Generically you need n initial values to fix a single solution of an n^{th} order ODE. There are exceptions to this rule though...

SEC 1.3 SLOPE FIELDS AND SOLUTION CURVES

Generic ODEs can be wild: almost none of them are solvable and even if they are solvable, their solutions can have strange behavior.

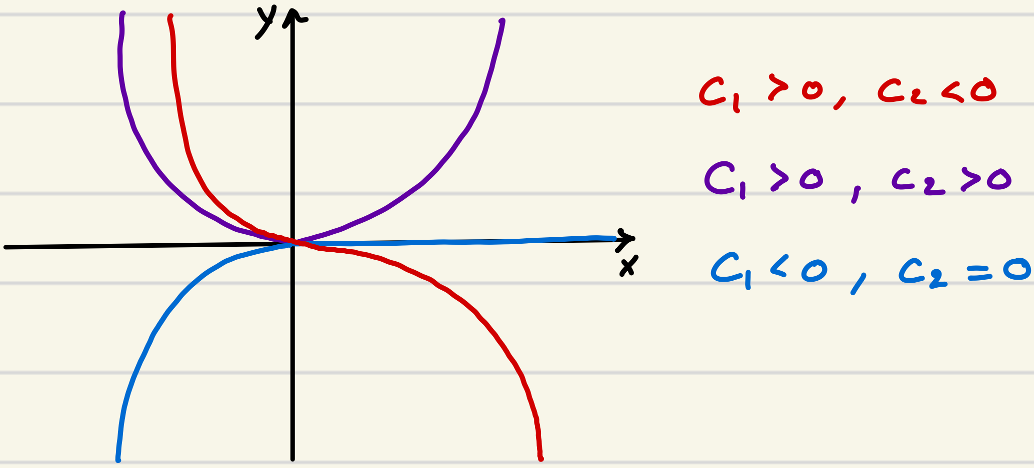
EXAMPLE

$$x y' = 2y$$

Later we will learn how to solve this equation and we'll find that its solutions are of the form

$$y = \begin{cases} C_1 x^2 & x<0 \\ 0 & x=0, \\ C_2 x^2 & x>0 \end{cases} \qquad c \underset{\text{element of}}{\in} \overset{\text{real numbers}}{\mathbb{R}}$$

The following plot shows several of these solutions:



Note that the number of solutions to the initial value problem $xy' = 2y,\ y(a) = b$ depends drastically on the values of a and b. If $a = 0$ then there are

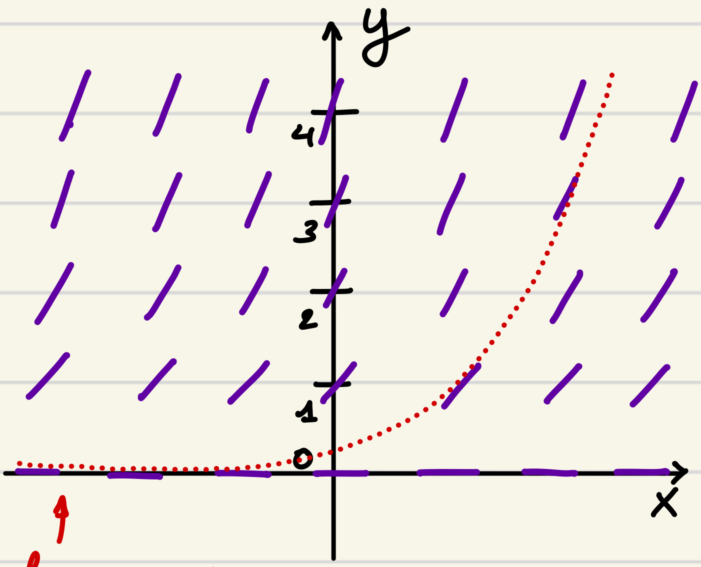
- there are no solutions if $b \neq 0$
- an ∞ number of solutions if $b = 0$

If $a \neq 0$ then, for any value of b, there are an ∞ number of solutions.

This pathological behavior arises despite the fact that the original DE looked very nice!

Often, we can't solve a DE in full generallity. In that case it can be useful to construct a so called "slope field" to get some insight into the behavior of the solutions.

EXAMPLE Construction of slope field for $y' = \sqrt{y}$



slopes must be tangent to solution curves

$y = 4 \qquad y' = \sqrt{4} = 2$

$y = 3 \qquad y' = \sqrt{3}$

$y = 2 \qquad y' = \sqrt{2}$

$y = 1 \Rightarrow y' = \sqrt{1} = 1$

$y = 0 \Rightarrow y' = \sqrt{0} = 0$

A slope field gives a nice general overview of the general behavior of the solutions to a DE. It is, however, not possible to provide answers for questions like:

- Is there a unique solution to the DE for given initial values?
- What is the limit of certain solutions as $x \to \pm\infty$?

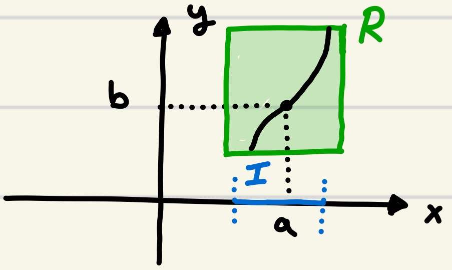
The problem is that a slope field only contains a finite amount of data.

There are more rigorous ways to obtain info about uniqueness of solutions to a first order IVP. Sadly (or interestingly !!?) general ODEs can be so wild so there are only pretty weak general theorems.

THEOREM EXISTENCE & UNIQUENESS OF SOLUTIONS TO 1ST ORDER ODEs.

Consider the IVP

$$⊛ \quad \begin{cases} y' = f(y,x) \\ y(a) = b \end{cases}$$



If f and $\frac{\partial f}{\partial y}$ are continuous on some rectangle R in the xy-plane that contains (a,b) in its interior then there exists an open interval I, that contains a, on which ⊛ has a unique solution.

EXAMPLES

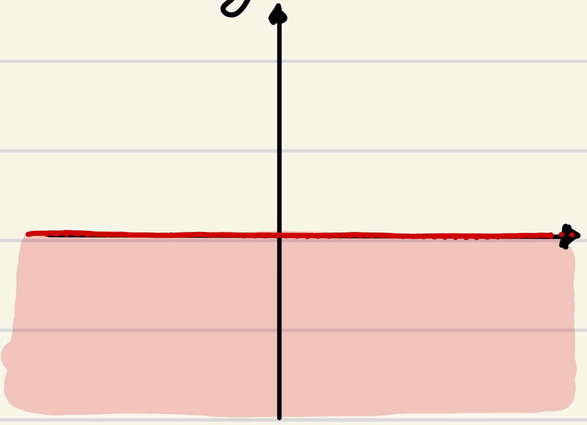
(1) For $y' = \sqrt{y}$ we have that

- $f(y,x) = \sqrt{y}$ which is continuous on the upper xy plane, i.e., for all (x,y) with $y \geq 0$
- $\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y}}$ which is continuous on the strict upper xy plane, i.e., for all (x,y) with $y > 0$

Therefore we can conclude that for any (a,b) with $b>0$ the IVP

$$\begin{cases} y' = \sqrt{y} \\ y(a) = b \end{cases}$$

has a unique solution on some open interval I of $\mathbb{R}$ to which a belongs



← Uniqueness & existence can't be guaranteed for any intial value belonging to lower half plane $(b \leq 0)$

(2) For the DE $y' = 1+y^2$ we have that $f(y,x) = 1+y^2$ so

- $f(y,x)$ is continuous everywhere
- $\frac{\partial f}{\partial y} = 2y$ is also continuous everywhere

so the IVP $\begin{cases} y' = 1+y^2 \\ y(a) = b \end{cases}$

has a unique solution on some open interval I of $\mathbb{R}$ to which a belongs.

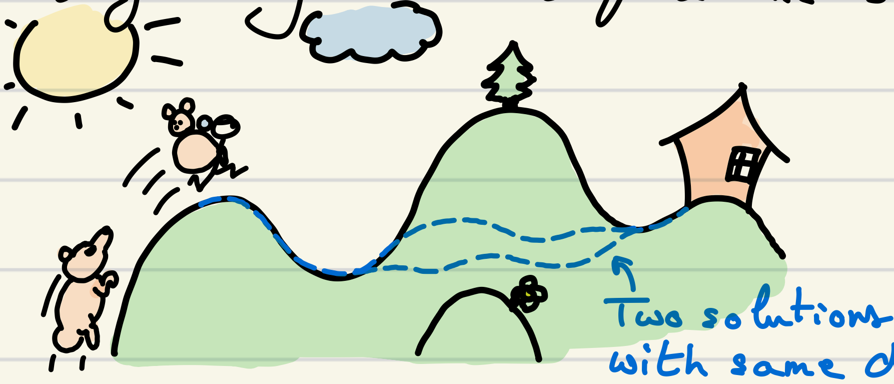
Note: I can be smaller than the width of the rectangle R for which the conditions of the theorem holds.

For $y' = 1 + y^2$ we will see that its solutions are given by $y = \tan(x+c)$. In particular: every solution has an ∞ number of points where it doesn't exist!

The theorem does not provide info on how big I is and is therefore quite weak. This is mainly because we want to say something about general 1st order ODEs. Later we will focuss on more $\underbrace{\text{specialized}}_{\text{linear ODEs}}$ cases for which there exist very strong results.

It is a common mistake to believe that DEs with pathological behavior have few applications.

EXAMPLE Consider a hiker being chased by a grizzly bear. To find the shortest way between 2 points on a non-flat surface the hiker has to solve a (system of) DE(s).



Depending on the initial position, such a system can have a unique, a finite number, or an infinite number of solutions.

CONVENTION

Often solutions to a DE have a smaller domain than $\mathbb{R}$ on which they are defined. When we write down solutions to a DE we will always demand that they are defind on as big a domain as possible.

EXAMPLE For $y' = 2\sqrt{y}$ we have the following solutions:

$$y = 0 \qquad (1)$$

$$y = (x-c)^2 \quad \text{for } x \geqslant c \qquad (2)$$

$$y = \begin{cases} 0 & \text{if } x < c \\ (x-c)^2 & \text{if } x \geqslant c \end{cases} \qquad (3)$$

While (2) is technically a solution to the DE, it is only a solution for $x \geqslant c$. Solution (2) is moreover "contained in" solution (3):

therefore we will disregard solutions of the form of (2) in favor of those of the form of (3). We say that the DE is solved by the functions

$$\boxed{y = 0 \qquad \text{and} \qquad y = \begin{cases} 0 & x < c \\ (x-c)^2 & x \geqslant c \end{cases}}$$

SEC 1.4. SEPERABLE EQUATIONS & APPLICATIONS

In sec 1.2. we saw that if a DE has the form

$$\frac{dy}{dx} = f(x)$$

that we can solve it by integrating both sides with respect to x. If

$$\frac{dy}{dx} = f(x,y)$$

we can't do this in general. There are some special cases, however, where a similar technique can be used. If $f(x,y) = \underbrace{g(x)}_{\text{function in x only}} \underbrace{h(y)}_{\text{function in y only}}$ then

$$\frac{dy}{dx} = g(x)\,h(y).$$

Lets assume $h(y) \neq 0$ for the moment. In that case we have that

$$\frac{1}{h(y)} \frac{dy}{dx} = g(x)$$

so

$$\int \frac{1}{h(y)} \frac{dy}{dx}\, dx = \int g(x)\, dx$$

By changeing the integration variable in to y we get (from the chain rule) that

$$\int \frac{dy}{h(y)} = \int g(x)\,dx$$

After evaluating both integrals we get

$$\boxed{H(y) = G(x) + C} \quad ⊛$$

where H & G satisfy $\frac{dH}{dy} = \frac{1}{h(y)}$ and $\frac{dG}{dx} = g(x)$.

⊛ is called the **general implicit solution** to the DE. It is not always possible to convert ⊛ to an explicit solution of the form

$$y = K(x) \quad \leftarrow \text{some function of } x.$$

Despite its name, the general solution does not always provide all solutions. Remember that we assumed that $h(y) \neq 0$ to obtain the general implicit solution. You can show (DO THIS AS AN EXERCISE!) that the functions $y = z$ where z is any constant for which $h(z) = 0$ are also solutions to the DE. Such solutions are called the **singular solutions**.

So the total set of solutions to $y' = g(x)\,h(y)$ is given by the general (implicit) solution together with the singular solutions.

Notes • If the DE is of the form $h(y)\frac{dy}{dx} = g(x)$ you can just integrate both sides and no singular solutions appear.

- Often the technique above is shortened as follows:

move h(x) over

$$\frac{dy}{dx} = g(x)\,h(y)$$

move dx over

$$\xrightarrow{\text{if } h(y)\neq 0} \quad \frac{dy}{h(y)} = g(x)\,dx$$

$$\Longrightarrow \quad \int \frac{dy}{h(y)} = \int g(x)\,dx$$

Despite the fact that this makes mathematicians' toes curl, I know everyone will use the shorter version and so will I.

EXAMPLES

(1) Solve $\frac{dy}{dx} = x\,y$

We start by finding the singular solutions. The DE is of the form

$$\frac{dy}{dx} = g(x)\,h(y) \quad \text{with } g(x) = x,\ h(y) = y$$

so we get a solution $y = z$ for each value z for which $h(z) = 0$, i.e. for which $z = 0$.

Therefore, there is 1 singular solution: $\boxed{y = 0.}$

The general solution can be found as follows:

$$\frac{dy}{dx} = x\,y$$

$$\overset{\text{since } y\neq 0}{\Longleftrightarrow} \quad \int \frac{dy}{y} = \int x\,dx$$

$$\Leftrightarrow \ln|y| = \frac{x^2}{2} + c_1$$

$$\Leftrightarrow e^{\ln|y|} = e^{\frac{x^2}{2}+c_1} = c_2 e^{\frac{x^2}{2}} \qquad (\text{where } c_2 = e^{c_1} > 0)$$

$$\Leftrightarrow |y| = c_2 e^{\frac{x^2}{2}}$$

$$\Leftrightarrow \boxed{y = \pm c_2 e^{\frac{x^2}{2}}}$$

So the solutions to $y' = xy$ are

$$y = 0 \quad \text{and} \quad y = \pm c_2 e^{\frac{x^2}{2}} \quad \text{with } c_2 > 0$$

These can all be combined in one solution, namely

$$\boxed{y = c e^{\frac{x^2}{2}}} \qquad \text{with } c \text{ any real number.}$$

Indeed, $c = \pm c_2$ can be any nonzero real number and if we allow $c = 0$ then we also get our singular solution for free ☺

(e) Solve $y \frac{dy}{dx} = -x$.

Here, we don't need to find singular solutions since we can just integrate both sides of the DE

$$y \frac{dy}{dx} = -x$$

$$\Leftrightarrow \int y \, dy = -\int x \, dx$$

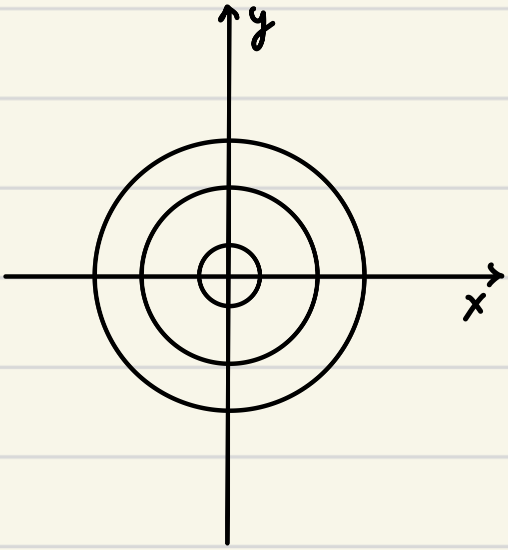
$$\Leftrightarrow \frac{y^2}{2} = -\frac{x^2}{2} + C_1$$

$$\Leftrightarrow y^2 = -x^2 + C_2 \quad \text{where } C_2 = 2C_1 \text{ is any real}$$

positive real number

In this case it is clearer to keep the solution in its implicit form and rearrange it as follows:

Let $c^2 = c_2$ then $\boxed{y^2 + x^2 = c^2}$ ← circles with radius c



The solutions in explicit form would be $y = \pm\sqrt{c^2 - x^2}$.

(3) Solve $\frac{dy}{dx} = y^2 - 1$

First we find the singular solutions. $g(x) = 1$ and $h(y) = y^2 - 1$

so $h(z) = 0 \iff z^2 - 1 = 0 \iff z = \pm 1$.

So there are 2 singular solutions: $y = -1$ & $y = +1$

Now we find the general solution:

$$\frac{dy}{dx} = y^2 - 1$$

$$\iff \int \frac{dy}{y^2 - 1} = \int dx$$

$$\iff \int \frac{1}{2}\left(\frac{1}{y-1} - \frac{1}{y+1}\right) dy = x + c_1 \qquad \text{with } c_1 \in \mathbb{R}$$

$$\iff \frac{1}{2} \ln|y-1| - \frac{1}{2} \ln|y+1| = x + c_1$$

$$\iff \ln\left|\frac{y-1}{y+1}\right| = 2x + 2c_1$$

$$\Leftrightarrow \left|\frac{y-1}{y+1}\right| = e^{2x+2c_1}$$

$$\Leftrightarrow \frac{y-1}{y+1} = \pm c_2 e^{2x} \qquad \text{with } c_2 = e^{2c_1} > 0$$

$$\Leftrightarrow y-1 = (y+1)c_3 e^{2x} \qquad \text{with } c_3 = \pm c_2 \neq 0$$

$$\Leftrightarrow y(1-ce^{2x}) = 1 + c_3 e^{2x}$$

$$\Leftrightarrow y = \frac{1+c_3 e^{2x}}{1-c_3 e^{2x}} \qquad \text{with } \bar{c} \neq 0$$

We can absorb 1 of our two singular solutions in the general one by allowing c_3 to be equal to 0. The other singular solution cannot be obtained this way so we find that all solutions to $y' = y^2-1$ are given by

$$\boxed{y = -1 \text{ and } y = \frac{1+ce^{2x}}{1-ce^{2x}} \text{ with } c \in \mathbb{R}}$$

(4) (Naive population model) Solve $\frac{dp}{dt} = \overset{\text{constant}}{k} p$

First we start with the singular solution. In this case $g(t) = k$ $h(p) = p$ so $h(z) = 0 \Leftrightarrow z = 0$ and therefore $p = 0$ is the only singular solution.

The general solution can be found as follows:

$$\frac{dp}{dt} = kp$$

$$\Leftrightarrow \int \frac{dp}{p} = \int k\,dt$$

$$\Leftrightarrow \ln|p| = kt + c_1$$

$$\Leftrightarrow |p| = e^{kt+c_1}$$

$$\Leftrightarrow p = \pm c_2 e^{kt} \qquad c_2 > 0$$

We can absorb the singular solution in the general one by allowing c_2 to be 0:

$$\boxed{p = c\,e^{kt}} \quad \text{with } c \in \mathbb{R}.$$

There are quite a lot of models in nature with exponential behavior, e.g.,

- Radioactive decay: $N(t) = N_0 e^{-\overset{k>0}{k}t}$
  ↑ number of molecules from a certain isotope
- Concentration of caffeine in your body: $\rho(t) = \rho_0 e^{-\overset{k>0}{k}t}$
  (as long as you don't refill)
  ↑ % of caffeine in blood = efficiency when playing whack-a-mole

Typically k is not given but a related quantity, the half-time τ, is given. τ is the amount of time it takes for a number satisfying $x = x_0 e^{-kt}$ to become $\frac{x_0}{2}$.

EXAMPLE The half-time of caffeine for the average human is 5h. This means that

$$\rho(5) = \frac{1}{2}\rho_0$$

but

$$\rho(5) = \rho_0 e^{5k}$$

so

$$\frac{1}{2}\cancel{\rho_0} = \cancel{\rho_0} e^{-5k} \qquad (\text{assuming } \rho_0 \neq 0)$$

$$\Leftrightarrow \quad e^{-5k} = \frac{1}{2}$$

$$\Leftrightarrow \quad k = -\frac{1}{5}\ln\left(\frac{1}{2}\right)$$

$$= \frac{1}{5}\ln(2)$$

If you drink a cup of coffe at 10 am, then at 10 pm you have

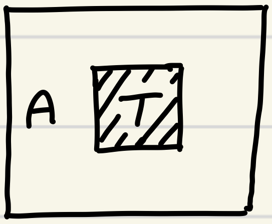
$$\begin{aligned}\rho(12) &= \rho_0\, e^{-\frac{1}{5}\ln(2)\cdot 12}\\ &= \rho_0 \left(e^{\ln(2)}\right)^{-\frac{12}{5}}\\ &= \rho_0\; 2^{-\frac{12}{5}}\\ &\approx \frac{\rho_0}{5}\end{aligned}$$

so 80% of the caffeine is gone.

A classic example of a seperable DE comes from Newton's law of cooling.

This law says that an object's temperature T, as a function of time t, is related to the surrounding temperature, A, as follows:

$$\frac{dT}{dt} = \overset{\text{constant}}{\overset{\downarrow}{k}}(A-T)$$



If we assume A is constant then this is a seperable DE with $g(t) = k$, $h(T) = A - T$. It has 1 singular solution: $\boxed{T = A}$ (body has same temp as its surroundings).

If $T \neq A$, then

$$\int \frac{dT}{A-T} = \int k\, dt$$

$$\Leftrightarrow -\ln|A-T| = kt + c_1$$

$$\Leftrightarrow |A-T| = e^{-kt-c_1} = \underbrace{(e^{-c_1})}_{\bar{c}} e^{-kt}$$

$$\Leftrightarrow A-T = \pm c_2 e^{-kt} \qquad \text{with } c_2 = e^{-c_1} > 0$$

$$\Leftrightarrow T = A \pm c_2 e^{-kt}$$

We can absorb the singular solution to get

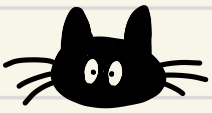
$$\boxed{T = A + c e^{-kt}} \quad \text{with } c \in \mathbb{R}$$

Depending on whether $T_0 > A$ or $T_0 < A$ the constant c will respectively be negative or positive.

EXAMPLE

You're having a B-day party at 4 p.m. and bought some cake a day ago which you put in the fridge at a temperature of 280 K. ← Kelvin = superior unit of temp ☺

You do some last minute shopping and return at 2 p.m. to find the cake lying and partly eaten on the floor Ö. There are 2 suspects:

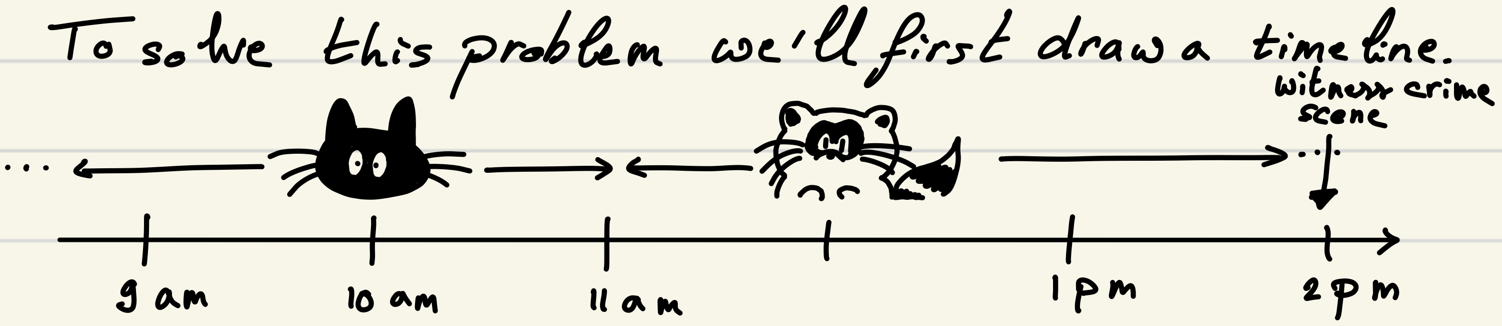


- Mrs. Whiskers
- At home until 11 a.m.
- Alibi from 11 a.m. onwards: sleeping on neighbours' dog house

- Mr. Scruffles
- Wakes up at 11 a.m.
- Alibi before 11 a.m.: laziest raccoon in the world. Can't get out of bed.

In order to find out who doesn't deserve dinner, you measure the temperature of the cake and find it is 290 K. Given that the room temperature is a constant 295 K and that, via measurement, you find that $k \approx \frac{\ln(3)}{4} \frac{1}{h}$ who is going to bed with an empty stomach? unit: 1 per hour.

To solve this problem we'll first draw a time line.



$T = 280$, $t = ?$ } → cake dropped from fridge

$T = 290$, $t = 0$

According to the law of cooling: $T = A + c\,e^{-kt}$, $A = 295$

At $t = 0$, $T = 290$ so

$$290 = 295 + c\,e^{-k \cdot 0} = 295 + c$$

$$\Rightarrow c = -5$$

So $$T = 295 - 5\,e^{-\frac{\ln(3)}{4}t}$$

therefore the moment $T = 280$ = fridge temp, we have that

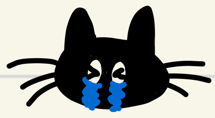
$$280 = 295 - 5\,e^{-\frac{\ln(3)}{4}t}$$

$$\Leftrightarrow -15 = -5\,e^{-\frac{\ln(3)}{4}t}$$

$$\Leftrightarrow \ln(3) = -\frac{\ln(3)}{4}t$$

$$\Leftrightarrow t = -4$$

So it was 2 p.m. − 4 h = 10 a.m. when the cake left the fridge.



SEC 1.5 LINEAR FIRST ORDER ODEs

In this section we turn our attention to ODEs of the form:

$$\overbrace{\frac{dy}{dx} + y\,P(x) = Q(x)}^{\text{linear } 1^{st} \text{ order ODE} = L1DE}$$

The idea for solving this equation is to bring it in a form where we, again, can integrate both sides of the equation without the need to know anything about y.
In particular, we want to write $y' + y\,P(x)$ in the form $\frac{d}{dx}(y\,p(x))$. This is not always possible, though. E.G.,

$\frac{dy}{dx} + y = x$ can not be written as $\frac{d}{dx}(y\,p(x)) = x$.

We can, however, always multiply the DE by a suitable $p(x)$ such that it takes the form $\frac{d}{dx}(y\,p(x)) = x\,p(x)$.

EXAMPLE If we multiply the equation

$$\frac{dy}{dx} + y = x$$

by $p(x) = e^x$, we get

$$\frac{dy}{dx}e^x + y\,e^x = x\,e^x$$

$$\Longleftrightarrow \frac{d}{dx}\left(y\,e^x\right) = x\,e^x$$

$$\Leftrightarrow \cancel{\int \frac{d}{dx}}(y e^x)\ \cancel{dx} = \int x e^x dx$$

$$\Leftrightarrow y = (x-1)e^x + C$$

$$\Leftrightarrow \boxed{y = x - 1 + c e^{-x}}$$

It turns out that for any L1DE of the form

$$\frac{dy}{dx} + y P(x) = Q(x)$$

$\rho(x) = e^{\int P(x) dx}$. Such a factor, $\rho(x)$, is called an *integration factor.*

Using an integration factor we can solve any L1DE:

$$\frac{dy}{dx} + y P(x) = Q(x)$$

$$\Leftrightarrow \frac{dy}{dx}\rho(x) + y P(x)\rho(x) = \rho(x) Q(x)$$

$$\Leftrightarrow \frac{d}{dx}\left(y \rho(x)\right) = \rho(x) Q(x)$$

$$\Leftrightarrow y \rho(x) = \int \rho(x) Q(x) dx$$

$$\Leftrightarrow y = \frac{1}{\rho(x)} \int \rho(x) Q(x) dx \quad \circledast$$

Note since $\rho(x)$ appears both in the numerator and denominator of $\circledast$, you don't need to care about the integration constant in $e^{\int P\,dx}$. These constants will always cancel.
The integration constant in $\int \rho(x)\,Q(x)\,dx$ still important, though!

So we have the following procedure for solving an L1DE:

(0) Find out what $P(x)$ is.
(1) Calculate $\rho(x) = e^{\int P(x)\,dx}$ (without integration const)
(2) Multiply the DE by $\rho(x)$
(3) Recognize the left-hand side as $\frac{d}{dx}(\rho(x)\,y)$
↳ if you don't recognize this, you might have made a mistake
(4) Integrate the equation
(5) Solve for y

EXAMPLES

(1) Solve $\quad y' - y = e^{-2x}$

(0) $P(x) = -1$

(1) $\rho(x) = e^{\int P(x)\,dx} = e^{-x}$

(2) $e^{-x}y' - e^{-x}y = e^{-3x}$

(3) $\frac{d}{dx}(e^{-x}y) = e^{-3x}$

(4) $e^{-x}y = -\frac{1}{3}e^{-3x} + c$

(5) $\boxed{y = -\frac{1}{3}e^{-2x} + ce^{x}}$

(2) Solve $(x^2+1)y' + 2xy = 6x$

First we divide the equation by x^2+1 to bring it in the standard form

$$y' + \frac{2x}{x^2+1}y = \frac{6x}{x^2+1}$$

(0) $P(x) = \frac{2x}{x^2+1}$

(1) $\int P(x)dx = \int \frac{2x}{x^2+1}dx = \ln|x^2+1| = \ln(x^2+1)$

so $\rho(x) = e^{\int p\,dx} = e^{\ln(x^2+1)} = x^2+1$

(2) $(x^2+1)y' + 2xy = 6x$

(3) $\frac{d}{dx}\left((x^2+1)y\right) = 6x$

(4) $(x^2+1)y = 3x^2 + c$

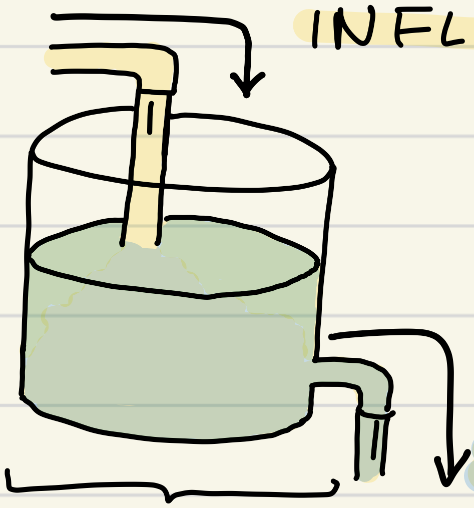
(5) $\boxed{y = \frac{3x^2+c}{x^2+1}}$

SEC 1.5. LINEAR FIRST ORDER ODEs

Previous lecture we showed how to solve

$$\frac{dy}{dx} + y\,P(x) = Q(x)$$

This lecture we'll setup the DEs for mixture problems, which are L1 DEs, and solve them. The mixture problems are of the following kind.



INFLOW at rate r_{in} (l/s) of liquid with concentration c_{in} of solute. Both r_{in}, c_{in} are assumed constant.

OUTFLOW at rate r_{out} (l/s) of liquid with concentration c_{out}

Tank that contains volume V of liquid with concentration

$$c = c_{out} = \frac{x}{V}$$

→ None of these are assumed constant.

The standard question for a mixture problem is: what is the amount of solute x in the tank as a function of time?

The DE for x can be set up as follows.
Let t, $t+\Delta t$ be two moments of time, close to each other.
Then we, approximately, have that at $t+\Delta t$

$$x(t+\Delta t) \approx x(t) + C_{in}\, r_{in}\, \Delta t - C_{out}(t)\, r_{out}(t)\, \Delta t$$

$$\Leftrightarrow \frac{x(t+\Delta t)-x(t)}{\Delta t} \approx C_{in}\, r_{in} - C_{out}(t)\, r_{out}(t)$$

This is only an approximation because C_{out} an r_{out} change over time and here we assumed that over the whole interval Δt their value remains the same as that at time t. If we let $\Delta t \to 0$, however, this approximation is exact and we get

$$\lim_{\Delta t \to 0} \frac{x(t+\Delta t)-x(t)}{\Delta t} = \frac{dx}{dt} = r_{in}\, C_{in} - r_{out}(t)\, C_{out}(t)$$

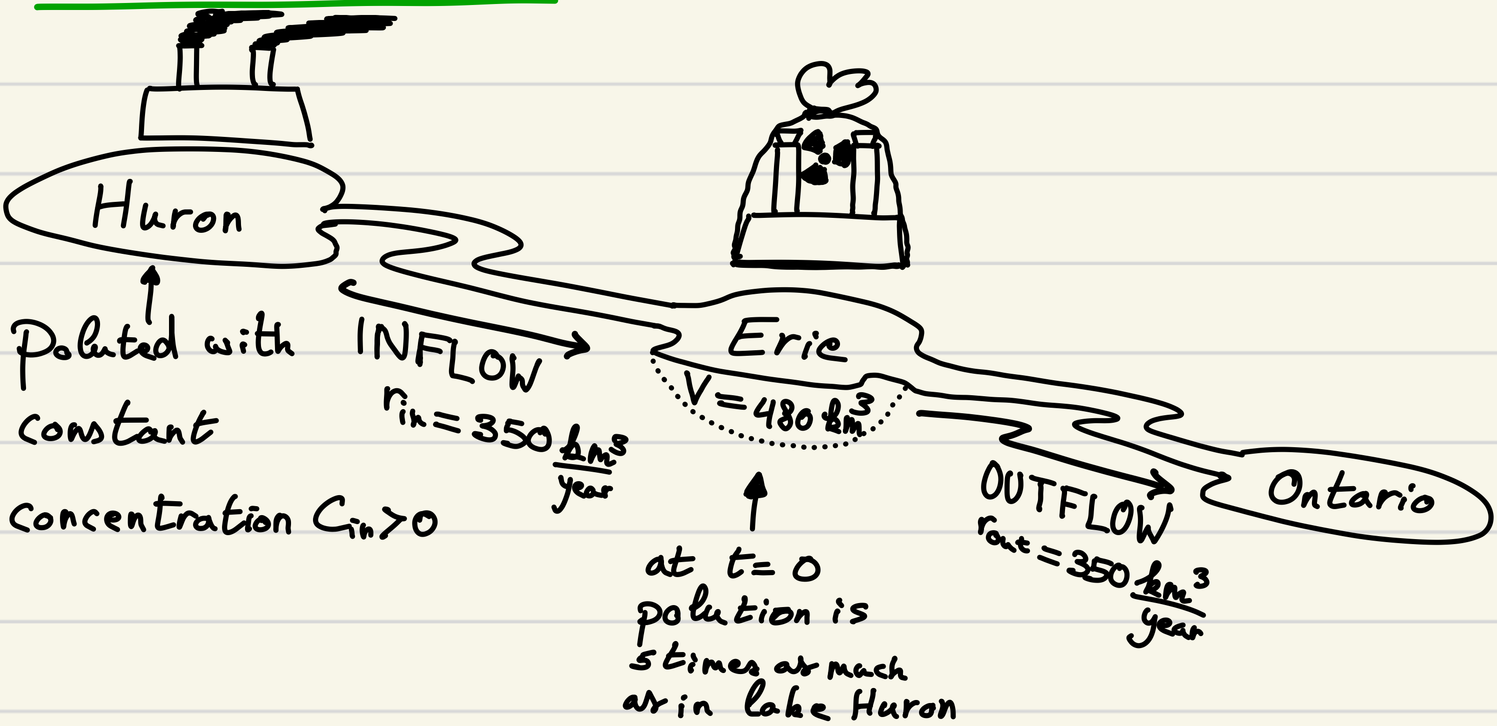
Since $C_{out} = \frac{x}{V}$ and $V = \overset{\substack{\text{volume at } t=0. \\ \downarrow}}{V_0} + (r_{in}-r_{out})t$ we find that

$$\frac{dx}{dt} = r_{in}\, C_{in} - \frac{x}{V_0 + (r_{in}-r_{out})t}$$

or

$$\boxed{\frac{dx}{dt} + x\underbrace{\frac{1}{V_0+(r_{in}-r_{out})t}}_{P(t)} = \underbrace{r_{in}\, C_{in}}_{Q(t)\,\leftarrow\,\text{constant}}}$$

EXAMPLES



QUESTION After closing the factories at lake Erie, how long before lake Erie is poluted only twice as much as lake Huron?

Solution

$$\frac{dx}{dt} + x \frac{r_{out}}{V_0 + t(r_{in} - r_{out})} = r_{in} C_{in}$$

Since $r_{in} = r_{out}$ we have that

$$\frac{dx}{dt} + \frac{r_{out}}{V_0} = r_{in} C_{in}$$

To avoid writing too many fractions we set $a = \overbrace{\frac{r_{out}}{V_0}}^{\text{constant}}$, $b = \overbrace{r_{in} C_{in}}^{\text{constant}}$ so

$$\frac{dx}{dt} + x a = b$$

(c) $P(t) = a$

(1) $p(t) = e^{\int a} = e^{at}$

(2) $\frac{dx}{dt} e^{at} + x a e^{at} = b e^{at}$

(3) $\frac{d(xe^{at})}{dt} = b e^{at}$

(4) $x e^{at} = \frac{b}{a} e^{at} + k_1$ ← we use k as a constant to avoid confusion.

(5) $\boxed{x = \frac{b}{a} + k_1 e^{-at}}$

Since we need to know what C_{out} is, rather than x, we divide the solution by V_0 to get

$$\frac{x}{V_0} = C_{out} = \frac{b}{a V_0} + \frac{k_1 e^{-at}}{V_0} = \frac{b}{aV_0} + k_2 e^{-at}$$

Since $a = \frac{r_{out}}{V_0}$ and $b = r_{in} C_{in}$ and $r_{out} = r_{in}$ we have that $\frac{b}{a V_0} = C_{in}$ so

$$C_{out} = C_{in} + k_2 e^{-at}$$

We know that at $t = 0$, $C_{out} = 5 C_{in}$ so

$$5 C_{in} = C_{in} + k_2 e^{-a \cdot 0} = C_{in} + k_2$$

so $k_2 = 4 C_{in}$ and therefore

$$\boxed{C_{out} = C_{in} + 4 C_{in} e^{-at}}$$

We want to know for which value of t, $C_{out} = 2 C_{in}$, so

$$2 C_{in} = C_{in} + 4 C_{in} e^{-at}$$

$$\overset{C_{in} \neq 0}{\iff} \quad \frac{1}{4} = e^{-at}$$

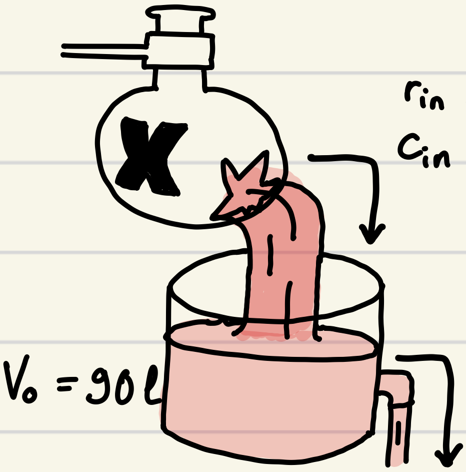
$$\iff \quad t = -\frac{1}{a} \ln\left(\frac{1}{4}\right) = \frac{V_0}{r_{out}} \ln(4) = \frac{48}{35} \ln(4) \approx \boxed{1.9}$$

Homework example

Professor Utonium made a liquid mixture of 90l of sugar, spice & everything nice which he starts pouring out at 3 l/s at time t=0.

At the same time he accidentally breaks the flask of !ELEMENT X!, containing a liquid mixture of sugar, spice & everything nice and !ELEMENT X! at a concentration of 2 kg/l, which flows into the tank at a rate of 4 l/s.

Given that the tank of 120l will explode when its maximum capacity is reached and that you need a concentration of 0.45 kg/l of !ELEMENT X! per powerpuff girl, how many powerpuff girls arrise from the process?



$$\left.\begin{aligned} r_{in} &= 4\ l/s \\ c_{in} &= 2\ kg/l \end{aligned}\right\} \quad r_{in}c_{in} = 8$$

$$\left. r_{out} = 3\ l/s \right\} \quad \frac{r_{out}}{V_0 + t(r_{in} - r_{out})} = \frac{3}{90+t}$$

The DE for this problem is

$$\frac{dX}{dt} + X\frac{3}{90+t} = 8$$

(0) $P(t) = \frac{3}{90+t}$

(1) $\int P(t)dt = 3\ln|90+t| = 3\ln(90+t)$ (since we only care about $t>0$)

so $\rho(t) = e^{\int P(t)dt} = (90+t)^3$

(2) $\frac{dX}{dt}(90+t)^3 + X\,3(90+t)^2 = 8(90+t)^3$

(3) $\frac{d}{dt}\left(X(90+t)^3\right) = 8(90+t)^3$

(4) $X(90+t)^3 = 8\int(90+t)^3 dt = \frac{8}{4}(90+t)^4 + k$

(5) $X = 2(90+t) + \frac{k}{(90+t)^3}$

At $t=0$ $X=0$ so

$$0 = 2\ 90 + \frac{k}{90^3}$$

$$\Leftrightarrow k = -2\cdot 90^4$$

so

$$X = 2(90+t) - \frac{2\ 90^4}{(90+t)^3}$$

The tank fills up at 1 l/s and has 120 l - 90 l = 30 l remaining space at $t=0$. It, therefore, fills up in 30 s. At that time

$$c = \frac{X}{V} = \frac{X}{120} = \frac{2}{120}(90+30) - \frac{2\ 90^4}{(90+30)^3}$$

$$= 2 - 2\frac{90^4}{120^3} = 2\left(1 - 90\cdot\left(\frac{3}{4}\right)^3\right)$$

repeated 6'es

$$= 1.3\overline{6} = \boxed{3}.\ 0.45 + 0.0\overline{1}$$

SEC 1.6. SUBSTITUTION METHODS & EXACT ODES

Sometimes a 1st order ODE is neither seperable, nor linear ← $y' + yP(x) = Q(x)$.

$\uparrow$ $y' = g(x)h(y)$

If this is the case, we might still be able to transform the ODE into a seperable or linear equation via substitution

EXAMPLE

$$\frac{dy}{dx} = (x+y+3)^2$$

Let $v = x+y+3$, and thus $y = x+3-v$,

then $\frac{dy}{dx} = \frac{d(x+3-v)}{dx} = 1 - \frac{dv}{dx}$.

The equation can be rewritten as

$$1 - \frac{dv}{dx} = v^2$$

$$\Leftrightarrow \quad \frac{dv}{dx} = 1 + v^2$$

$$\Leftrightarrow \quad \int \frac{dv}{1+v^2} = \int 1 \, dx$$

$$\Leftrightarrow \quad \tan^{-1}(v) = x + C$$

$$\Rightarrow \quad v = \tan(x+c)$$

$$\Leftrightarrow \quad y + x + 3 = \tan(x+c)$$

$$\Leftrightarrow \quad y = \tan(x+c) - x - 3$$

Note When substituting some function $f(x,y)$ for a new unknown function v, one also needs to replace $\frac{dy}{dx}$ by a function that only contains $\frac{dv}{dx}$, v, and x.
If $v = f(x,y)$, we do this by solving $v=f(x,y)$ for y, i.e. rewriting it as $y = g(v,x)$ for some function g of v and x and then calculating $\frac{dy}{dx} = \frac{dg}{dx}$ where you should regard v as a function of x. So don't forget the chain rule!

EXAMPLES

(1) For $\frac{dy}{dx} = \sin(x-y)$ you might want to set $v = x-y$.
Solving for y gives us $y = x - v$ and thus $\frac{dy}{dx} = 1 - \frac{dv}{dx}$

(2) For $\frac{dy}{dx} = e^{\frac{y}{x}}$ you might want to set $v = \frac{y}{x}$.
In that case $y = xv$ and thus $\frac{dy}{dx} = \frac{d}{dx}(xv) = v + x\frac{dv}{dx}$

There are several general classes of equations that can be transformed into seperable and/or linear equations by choosing the right substitution.

DEFINITION A homogeneous ODE is of the form $\frac{dy}{dx} = f(\frac{y}{x})$

EXAMPLES

(1) $\frac{dy}{dx} = \sin\left(\frac{y}{x}\right)$

(2) $\frac{dy}{dx} = 2\frac{x}{y} + \frac{3}{2}\frac{y}{x} = 2\frac{1}{\left(\frac{y}{x}\right)} + \frac{3}{2}\left(\frac{y}{x}\right)$

(3) $\frac{dy}{dx} = \tan\left(\sin\left(\sqrt{1+\left(\frac{y}{x}\right)^{164}}\right)\right) + \left(\frac{y}{x}\right)^{100000}$

A homogeneous DE begs for the substitution $v = \frac{y}{x}$.

Indeed, in that case $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and, therefore,

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

becomes

$$v + x\frac{dv}{dx} = f(v)$$

so

$$\frac{dv}{dx} = \frac{1}{x}\left(f(v) - v\right)$$

which is seperable.

EXAMPLE Solve $x\frac{dy}{dx} = y + \sqrt{x^2 - y^2}$ for $x \geqslant y > 0$

First we divide by x to get

$$\frac{dy}{dx} = \left(\frac{y}{x}\right) + \sqrt{1 - \left(\frac{y^2}{x^2}\right)}$$

Setting $v = \frac{y}{x}$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$:

$$v + x\frac{dv}{dx} = v + \sqrt{1-v^2}$$

$$\Longleftrightarrow \quad \frac{dv}{dx} = \frac{1}{x}\sqrt{1-v^2}$$

$$\Leftrightarrow \int \frac{dv}{\sqrt{1-v^2}} = \int \frac{dx}{x}$$

$$\Leftrightarrow \sin^{-1}(v) = \ln|x| + c$$

$$\overset{x>0}{\Longleftrightarrow} \quad v = \sin(\ln(x) + c)$$

$$\Leftrightarrow \quad y = x \sin(\ln(x) + c)$$

Now we introduce a second class of equations.

DEFINITION A Bernouilli equation is of the form

$$\frac{dy}{dx} + y P(x) = Q(x) y^n \quad \text{with } n \in \mathbb{R}$$

Note If $n = 0$ or $n = 1$ this is a linear equation.

A Bernouilli equation can be solved by substituting $v = y^{1-n}$, so $y = v^{\frac{1}{1-n}}$ and $\frac{dy}{dx} = \frac{1}{1-n} v^{\frac{n}{1-n}} \frac{dv}{dx}$.

This substitution transforms

$$\frac{dy}{dx} + y P(x) = Q(x) y^n$$

into

$$\frac{1}{1-n} v^{\frac{n}{1-n}} \frac{dv}{dx} + v^{\frac{1}{1-n}} P(x) = Q(x) v^{\frac{n}{1-n}}$$

$$\Leftrightarrow \quad \frac{dv}{dx} + (1-n) \frac{v^{\frac{1}{1-n}}}{v^{\frac{n}{1-n}}} P(x) = Q(x)(1-n)$$

$$\Leftrightarrow \quad \frac{dv}{dx} + (1-n) v^{\frac{1}{1-n} - \frac{n}{1-n}} P(x) = Q(x)(1-n)$$

$$\Leftrightarrow \quad \frac{dv}{dx} + v(1-n) P(x) = Q(x)(1-n)$$

which is a L1DE.

Note There are various subtle issues when substituting $v = y^{1-n}$. If $n > 1$ then y should be assumed to be $\neq 0$. If n is not an integer then the equation only makes sense for $y > 0$.
Often the exercises will ignore those subtleties. We will make sure that you don't have to worry about this on the exam by adding assumptions on the domains of y and x.

EXAMPLES

(1) Solve $\quad x \frac{dy}{dx} + 6y = 3x\, y^{\frac{4}{3}} \quad$ where $y > 0$, $x > 0$.

Since $x \neq 0$ we can divide the equation by x to get

$$\frac{dy}{dx} + y\left(\frac{6}{x}\right) = 3\, y^{4/3}$$

This is a Bernoulli equation with $n = \frac{4}{3}$.
The substitution $v = y^{1-\frac{4}{3}} = y^{-\frac{1}{3}}$ should do the trick.
If $v = y^{-\frac{1}{3}}$ then $y = v^{-3}$ so $\frac{dy}{dx} = -3\, v^{-4} \frac{dv}{dx}$ and our equation becomes

$$-3\, v^{-4} \frac{dv}{dx} + v^{-3}\left(\frac{6}{x}\right) = 3\, v^{-4}$$

$$\Longleftrightarrow \quad \frac{dv}{dx} + v\left(\frac{-2}{x}\right) = -1$$

This is a linear equation with

(0) $P(x) = \frac{-2}{X}$

(1) $\rho(x) = e^{\int -\frac{2}{x}} = e^{-2\ln|x|} = |x|^{-2} = x^{-2}$

(2) $\frac{dv}{dx} x^{-2} + v\left(\frac{-2}{x^{-3}}\right) = -x^{-2}$

(3) $\frac{d}{dx}\left(v\, x^{-2}\right) = - x^{-2}$

(4) $v\, x^{-2} = \frac{1}{X} + c$

(5) $v = x + cx^2$

Therefore $y^{-1/3} = x + cx^2$ so $y = (x + cx^2)^{-3}$

SEC 1.6. EXACT ODEs

For some DEs, such as separable DEs, the solution will be obtained in an implicit form:

$$F(x, y(x)) = C$$

From such a solution we can recover the original differential equation by deriving the equation with respect to x.

#partial derivative!

Remember: to take the derivative of a function with multiple arguments say, e.g.,

$$\frac{d}{dx}G(f_1, f_2, \dots, f_n)$$

we must assume each argument depends only on x. The formula for the derivative is then

$$\frac{dG}{dx}(f_1, \dots, f_n) = \frac{\partial G}{\partial f_1}\frac{df_1}{dx} + \frac{\partial G}{\partial f_2}\frac{df_2}{dx} + \dots + \frac{\partial G}{\partial f_n}\frac{df_n}{dx}$$

By applying this formula to $F(x,y(x))=C$ we get

$$\frac{d}{dx}\left(F(x,y(x))\right) = \frac{d}{dx}(C)$$

$$\Rightarrow \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Let $M(x,y)=\frac{\partial F}{\partial x}$, $N(x,y)=\frac{\partial F}{\partial y}$ then this becomes

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

or, more symmetrical:

$$\underbrace{\boxed{\overbrace{M(x,y)\,dx + N(x,y)\,dy}^{\text{Differential form}} = 0}}_{\text{Equation in differential form.}}$$

In retrospect, we see that if we are given a DE of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

or, equivalently,

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

and there exists a function

$$F(x,y)$$

such that

$$M(x,y) = \frac{\partial F}{\partial x}, \quad N(x,y) = \frac{\partial F}{\partial y}$$

then we can rewrite this DE as

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$$

or

$$d(F(x,y)) = 0$$

from which it follows that

$$F(x,y) = \overset{\text{constant}}{C}$$

and we have solved the DE.

DEFINITION If a differential form $M(x,y)\,dx + N(x,y)\,dy$ can be written as $dF(x,y)$ for some function $F(x,y)$ then we say that form (and also the equation $Mdx+Ndy=0$) is **exact**.

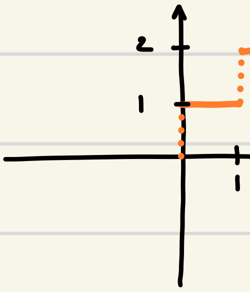
Note: not every differential form is exact.

EXAMPLE

Solve
$$\underbrace{2xy}_{M}\,dx + \underbrace{x^2}_{N}\,dy = 0$$

Note that this DE is exact since for $F(x,y) = x^2 \cdot y$ we have that

$$\frac{\partial F}{\partial x} = 2xy = M \text{ and } \frac{\partial F}{\partial y} = x^2 = N$$



So the equation can be written as

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$$

or
$$dF(x,y) = 0$$

from which it follows that

i.e.

$$F(x,y) = c$$

so
$$x^2 y = c$$

$$\Rightarrow \boxed{y = \frac{c}{x^2}} \quad \text{for } x \neq 0.$$

In this example F was given. One may wonder for general examples whether

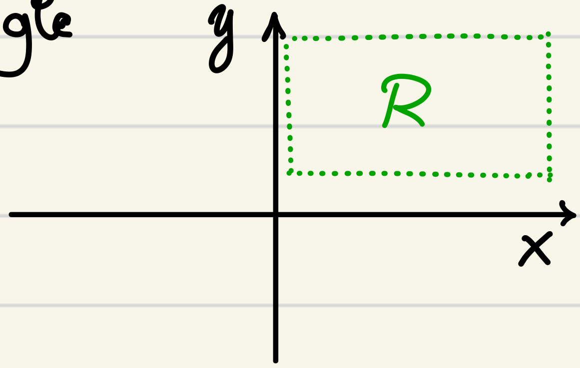
(a) there exists such an F?

(b) how to find such an F?

Question (a) is answered by the following theorem

THEOREM (EXACTNESS)

Consider an open rectangle R in the xy plane.



If $M, N, \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$ are continuous on R then

$$M\,dx + N\,dy = 0$$

is exact if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ on R.

EXAMPLE (Midterm 1 2023)

Find the value of b for which

$$(y\cos(8xy) + b\,x)dx + (b\,x\cos(8xy) - 2y)dy = 0$$

is exact and then solve that equation.

SOLUTION For any value of b we have that

$$M, N, \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$$

are continuous on $\mathbb{R}^2$.

So we need to have $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for all x,y.

$$\frac{\partial M}{\partial y} = \frac{\partial(y\cos(3xy) + bx)}{\partial y}$$

$$= \cos(3xy) - 3xy\sin(xy)$$

$$\frac{\partial N}{\partial x} = \frac{\partial(bx\cos(3xy) - 2y)}{\partial x}$$

$$= b\cos(3xy) - 3xyb\sin(3xy)$$

So

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \cos(3xy) - 3xy\sin(3xy) = b(\cos(3xy) - 3xy\sin(3xy))$$

$$\Rightarrow b = 1$$

How do we solve this equation? Or with other words: how do we find F?

<u>SOLUTION</u>: To find F we can use the following steps.

1) From $\frac{\partial F}{\partial x} = M(x,y)$ we see that F must be any function whose partial derivative w.r.t. x is $M(x,y)$.
Such a function can be found by integrating

$M(x,y)$ w.r.t. x where we assume that y is a constant.

Note that if $G(x,y)$ is a function with $\frac{\partial G}{\partial x} = M$ then $G(x,y) + g(y)$ also has this property.

So

$$F(x,y) = \int \overset{\text{assuming } y \text{ is constant}}{M(x,y)\,dx}$$
$$= G(x,y) + g(y)$$

2) To figure out what g is we can use the fact that

$$\frac{\partial F}{\partial y} = N(x,y)$$

so

$$\frac{\partial G(x,y)}{\partial y} + \underbrace{\frac{\partial g}{\partial y}}_{= \frac{dg}{dy} \text{ since } g \text{ only depends on } y} = N(x,y)$$

or

$$\frac{dg}{dy} = \underbrace{N(x,y) - \frac{\partial G(x,y)}{\partial y}}_{\text{Note that after doing this subtraction, there should be no occurence of } x \text{ anymore!}}$$

so

$$g = \underbrace{\int}_{\text{Normal integral}} \left(N(x,y) - \frac{\partial G}{\partial y}(x,y)\right) dy$$

Back to our example: we want to solve

$$(y\cos(3xy)+x)\,dx + (x\cos(3xy)-2y)\,dy = 0$$

where y is assumed const

1) $F = \int M(x,y)\,dx = \int (y\cos(3xy)+x)\,dx$

$$= \frac{\sin(3xy)}{3} + \frac{x^2}{2} + g(y)$$

2) $\frac{\partial F}{\partial y} = N$ so $\quad x\,\cancel{\cos(3xy)} + \frac{dg}{dy} = x\,\cancel{\cos(3xy)} - 2y$

or $\frac{dg}{dy} = -2y \Rightarrow g = -y^2 + c'$

Therefore $\quad F = \frac{\sin(3xy)}{3} + \frac{x^2}{2} - y^2 + c'$

and the solution to the DE is

$$F = c''$$

i.e. $\frac{\sin(3xy)}{3} + \frac{x^2}{2} - y^2 + c' = c''$

or, if we set $c = c'' - c'$ then

$$\boxed{\frac{\sin(3xy)}{3} + \frac{x^2}{2} - y^2 = c}$$

And now for something completely different...

SECOND DEGREE REDUCIBLE ODEs

DEFINITION A second degree ODE is reducible if it is of one of the two following forms

① $F(y, y', y'') = 0$ (contains no x)

② $F(x, y', y'') = 0$ (contains no y)

Both cases can be solved by reduction to a first order ODE.

CASE ① : Here we let y play the role of the independent variable and set

$$D = \frac{dy}{dx}$$

Our equation then takes the form

$$F\left(y, D, \frac{dD}{dx}\right) = 0$$

This equation is problematic since it still contains a derivative w.r.t. x and we only want to have y as an independent variable.

By using the chain rule we can transform $\frac{dD}{dx}$ as follows:

$$\frac{dD}{dx} = \frac{dD}{dy}\frac{dy}{dx} = \frac{dD}{dy}\cdot D$$

So our equation becomes

$$F\left(y, D, \frac{dD}{dy}\cdot D\right) = 0$$

Once this equation is solved, say by $D = f(y)$, then we can find y as a function of x by solving

$$D = \boxed{\frac{dy}{dx} = f(y)}$$

Note: since this equation contains an integration constant from solving the DE for D, you will have 2 different integration constants in your final solution. This is not surprising since the original DE was of the second order.

CASE ②. This case is simpler since we will keep x as our variable and set $D = y'$. Our DE then becomes

$$F\left(x, D, \frac{dD}{dx}\right) = 0$$

If $D = f(x)$ is a solution to this DE

then y as a function of x can be found by solving

$$D = \boxed{\frac{dy}{dx} = f(x)}$$

CHAPTER 2
MATHEMATICAL MODELS AND NUMERICAL METHODS

SEC 2.1. POPULATION MODELS

Population growth can be naively modeled as follows:

$$\frac{dP}{dt} = kP$$

This is an unrealistic model since any nonzero solution $P = ce^{kt}$ has $P \to \infty$ as $t \to \infty$. This model does not take account of the fact that the birth & death rate of a population are nonconstant.

Let

- $\beta(t)$ be the birth rate $= \dfrac{\#\text{ births}}{(\text{time unit})(\text{population unit})}$
- $\delta(t)$ be the death rate $= \dfrac{\#\text{ deaths}}{(\text{time unit})(\text{population unit})}$

then we have that

$$P(t+\Delta t) \approx P(t) + \beta(t)P(t)\Delta t - \delta(t)P(t)\Delta t$$

or

$$\frac{P(t+\Delta t) - P(t)}{\Delta t} \approx (\beta(t) - \delta(t))P(t)$$

As $\Delta t \to 0$ this approximation (which occurs because we use the values of β & δ at time t while between t and Δt these values change) becomes an exact equality.

Therefore:

$$\lim_{\Delta t \to 0} \frac{P(t+\Delta t) - P(t)}{\Delta t} = \frac{dP}{dt} = (\beta(t) - \delta(t))P(t)$$

<u>Note</u>: if we assume that β and δ are constant then we retrieve our naive model with $k = \beta - \delta$.

In reality one often finds that the birth rate decreases and the death rate increases as a function of the population.

One of the simplest ways to incorporate a decreasing birth/death ratio is by setting

$$\beta(t) - \delta(t) = a - bP(t)$$

In that case we have that

$$\frac{dp}{dt} = (a - bP(t))P(t)$$
$$= b\left(\frac{a}{b} - P(t)\right)P(t)$$

Let $k = b$, $M = \frac{a}{b}$ then:

$$\boxed{\frac{dp}{dt} = k(M - P(t))P(t)} \quad \circledast$$

M is called the carrying capacity:

- if $P > M$ then $\frac{dp}{dt} < 0 \Rightarrow$ population decreases
- if $0 < p < M$ then $\frac{dp}{dt} > 0 \Rightarrow$ population increases

Since $\circledast$ is a seperable DE, we can solve it.

Step 1 Assume $k(M-P)P = 0$. Then $P = 0$ and $P = M$ solve the DE.

Step 2 Assume that $k(M-P)P \neq 0$

Then:

$$\circledast \Rightarrow \int \frac{dp}{(M-P)P} = \int k\,dt$$

Lets write $\frac{1}{(M-P)P}$ as $\frac{A}{M-P} + \frac{B}{P} = (**)$

$$(**) = \frac{AP + BM - BP}{(M-P)P} = \frac{P(A-B) + MB}{(M-P)\,P}$$

So $\begin{cases} A - B = 0 \\ M B = 1 \end{cases} \Longrightarrow \begin{matrix} B = 1/M \\ A = 1/M \end{matrix}$

$$\Rightarrow \frac{1}{M}\int\left(\frac{1}{M-P} + \frac{1}{P}\right)dP = kt + C'''$$

$$\Rightarrow -\ln|M-P| + \ln|P| = Mkt + C''$$

$$\Rightarrow \ln\left|\frac{P}{M-P}\right| = Mkt + C''$$

useful for finding k and/or C'

$$\Rightarrow \left|\frac{P}{M-P}\right| = e^{C''}e^{Mkt}$$

$$\Rightarrow \frac{P}{M-P} = \pm e^{C''}e^{Mkt}$$

$$\Rightarrow \frac{P}{M-P} = C'e^{Mkt} \qquad C \neq 0$$

$$\Rightarrow P = MC'e^{Mkt} - PC'e^{Mkt}$$

$$\Rightarrow P(1 + C'e^{Mkt}) = MC'e^{Mkt}$$

$$\Rightarrow P = \frac{MC'e^{Mkt}}{1 + C'e^{Mkt}}$$

Let $C = \frac{1}{C'}$ and divide numerator & denominator by $C'e^{Mkt}$ to get

$$P = \frac{M}{1 + Ce^{-Mkt}}$$

If at $t = 0$, $P = P_0$ $\leftarrow$ $\neq 0$ and $\neq M$ then we find that

$$P_0 = \frac{M}{1 + C} \Rightarrow 1 + C = \frac{M}{P_0} \Rightarrow C = \frac{M}{P_0} - 1$$

$$\Rightarrow C = \frac{M - P_0}{P_0}$$

so $$P = \frac{M}{1 + \frac{(M - P_0)}{P_0}e^{-Mkt}} = \frac{MP_0}{P_0 + (M - P_0)e^{-Mkt}}$$

If we allow P_0 to be 0 or M then we have combined our solutions from step 1 with those of step 2.

EXAMPLES (1) Purdue has 15100 members of staff and 41573 students. 200 of these students are bio-engineers. At the start of the semester an experiment in the bio labs goes horribly wrong and all 200 bio engineers become zombies. To protect the rest of the world, a wall is built around Purdue so no one can leave/enter. Let H be the population of humans and Z be the population of zombies. The chance that a single zombie encounters and infects a human is proportional to the number of humans. If we assume that no humans and zombies die and no humans are born then

$$\frac{dZ}{dt} = kZH$$

Since $Z+H$ is constant, say $Z+H=M$ we have that

$$\frac{dZ}{dt} = kZ(M-Z)$$

Now, if after 7 days we count 800 zombies then, how long will it take before half the population of Purdue actually craves big brains.

We know that

$$Z = \frac{Z_0 M}{Z_0 + (M - Z_0)e^{-Mkt}}$$

with $M = 15100 + 41573 = 57673$

$Z_0 = 200$

$\Rightarrow$ $M - Z_0 = 57473$

$MZ_0 = 115346$

In order to solve the problem we need to find k

Let $Z_{t_1} = Z$ at $t = t_1$ then

$$Z_{t_1} = \frac{Z_0 M}{Z_0 + (M - Z_0)e^{-kMt_1}}$$

$$\Rightarrow Z_{t_1} Z_0 + Z_{t_1}(M - Z_0)e^{-kMt_1} = Z_0 M$$

$$\Rightarrow Z_{t_1}(M - Z_0)\, e^{-kMt_1} = Z_0(M - Z_{t_1})$$

$$\Rightarrow e^{-kMt_1} = \frac{Z_0}{Z_{t_1}} \frac{M - Z_{t_1}}{M - Z_0}$$

$$\Rightarrow -kMt_1 = \ln\left(\frac{Z_0}{Z_{t_1}} \frac{M - Z_{t_1}}{M - Z_0}\right)$$

$$\Rightarrow k = \frac{1}{Mt_1} \ln\left(\frac{Z_{t_1}}{Z_0} \frac{M - Z_0}{M - Z_{t_1}}\right) \qquad ✪$$

In our case: at $t = 7$ $Z_7 = 800$

so $k = \dfrac{1}{7 \cdot 57673} \ln\left(\dfrac{800}{200} \dfrac{57473}{56862}\right) \approx 3.52 \cdot 10^{-6}$

When is $Z = \frac{M}{2}$?

Rather than filling this data into the original equation and solving for t we can use equation ⊛:

$$t = \frac{1}{Mk}\ln\left(\frac{M}{2Z_0}\,\frac{M-Z_0}{M-\frac{M}{2}}\right)$$

$$= \frac{1}{Mk}\ln\left(\frac{M-Z_0}{Z_0}\right)$$

$$\approx 28$$

(2) The raccoon whisperer moves into town and brings with him 40 socially inept raccoons whose mating ritual depends solely on randomly bumping into one another. It is reasonable to say that the number of such encounters per unit of time is proportional to

$$\underset{\#\,\text{♂ raccoons}}{\frac{P}{2}} \cdot \underset{\#\,\text{♀ raccoons}}{\frac{P}{2}}$$

so $\beta(t) = dP$ (not P^2 since β is defined per unit of population)

If we assume the death rate is constant

$$\delta(t) = b$$

then the population model is the following:

$$\frac{dp}{dt} = (\beta(t) - \delta(t))P$$
$$= (aP - b)P$$
$$= aP^2 - bP$$

Now, if in the first year 40 raccoons are born and 20 raccoons died, then how long does it take before the universe is filled with raccoons?

<u>SOLUTION</u> The birth rate is

$$\frac{40 \text{ raccoon}}{(1 \text{ year})(40 \text{ raccoon})} \approx \beta(t_0)$$

we will assume equality

start with 40

when $P(t_0)=40$. Since $\beta(t) = aP(t)$,

$$a = \frac{\beta(t)}{P(t)} \Rightarrow \boxed{a = \frac{1}{40}}$$

Likewise for b: the death rate at $t = t_0$ is

$$\frac{20 \text{ raccoon}}{(1 \text{ year})(40 \text{ raccoon})} = \delta(t_0) = b \quad \text{so } \boxed{b = \frac{1}{2}}$$

Our model becomes

$$\frac{dp}{dt} = \frac{1}{40}P^2 - \frac{1}{2}P$$
$$= \frac{1}{40}P(P-20) = \left(\frac{-1}{40}\right)P(20-P)$$

This is the same model as that for the zombies but with a negative value for k. We can recycle our solution:

$$P = \frac{P_0 M}{P_0 + (M - P_0)e^{-kMt}}$$

$$= \frac{40 \cdot 20}{40 + (20-40)\, e^{-\left(\frac{-1}{40}\right) \cdot 20t}}$$

$$= \frac{40}{2 - e^{t/2}}$$

So $\lim\limits_{t \to 2\ln(2)} P(t) = \infty$ so it takes $2\ln(2)$ years before the universe is filled with raccoons.

Some things to think about:

1) What makes this model so unrealistic.
2) Is our way of calculating the constants a and b fair, given that we assumed a constant β for a whole year?
3) Is it even appropriate to model a discrete population of size 40 by using a DE?

SEC 2.2. EQUILIBRIA AND BIFURCATIONS

When solving a seperable equation

$$\frac{dy}{dx} = g(x)h(y) \quad ⊛$$

we always checked for what constant values of y, $h(y)$ is 0. These constant values provide singular solutions to ⊛, which are also called equilibria.

If $g(x) = 1$ then we can obtain a lot of qualitative information about the solutions of ⊛ by examining the function $h(y)$.

EXAMPLE For the logistic model

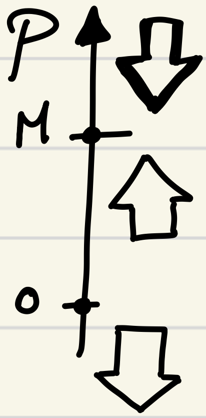
$$\frac{dP}{dt} = kP(M-P)$$

We see that $P=0$ and $P=M$ are equilibria. If $0<P<M$ then $\frac{dP}{dt}>0$ so P will increase. If $P>M$ then $\frac{dP}{dt}<0$ so P will decrease.

(If P stands for a population then we assume $P \geq 0$. If P were an abstract function then for $P<0$ we would find that P decreases as well)

So the qualitative behavior of our solutions is determined by the sign of $\frac{dP}{dt} = ($sign of $kP(M-P))$.

Often one denotes the behavior of P using an axis:



	$\frac{dP}{dt}$
$P > M$	$-$
$P = M$	0
$0 < P < M$	$+$
$P = 0$	0
$P < 0$	$-$

DEFINITION We say that an equilibrium $y = e$ is

- **Stable** if it attracts solutions
- **Unstable** if it repels solutions
- **Semistable** if it both attracts and repels solutions

In our example M is a stable equilibrium: if you start with an intial value that is slightly above or slightly below M then it will converge to M as $t \to \infty$. 0, on the other hand, is an unstable equilibrium: if you start with an intial value that is close to 0, then it will diverge from 0 as $t \to \infty$.

The general procedure to determine equilibria of a DE

$$y' = h(y)$$

(where h is assumed to be continuous) and their properties goes as follows.

1) Determine all values of y for which $h(y) = 0$.
2) Determine the sign of $h(y)$ for values of y different than the equilibria. Since h is assumed continuous we can find the sign of $h(y)$ between two equilibria (say $y = e_1$ and $y = e_2$) by plugging in a value of y between e_1 and e_2 in $h(y)$.

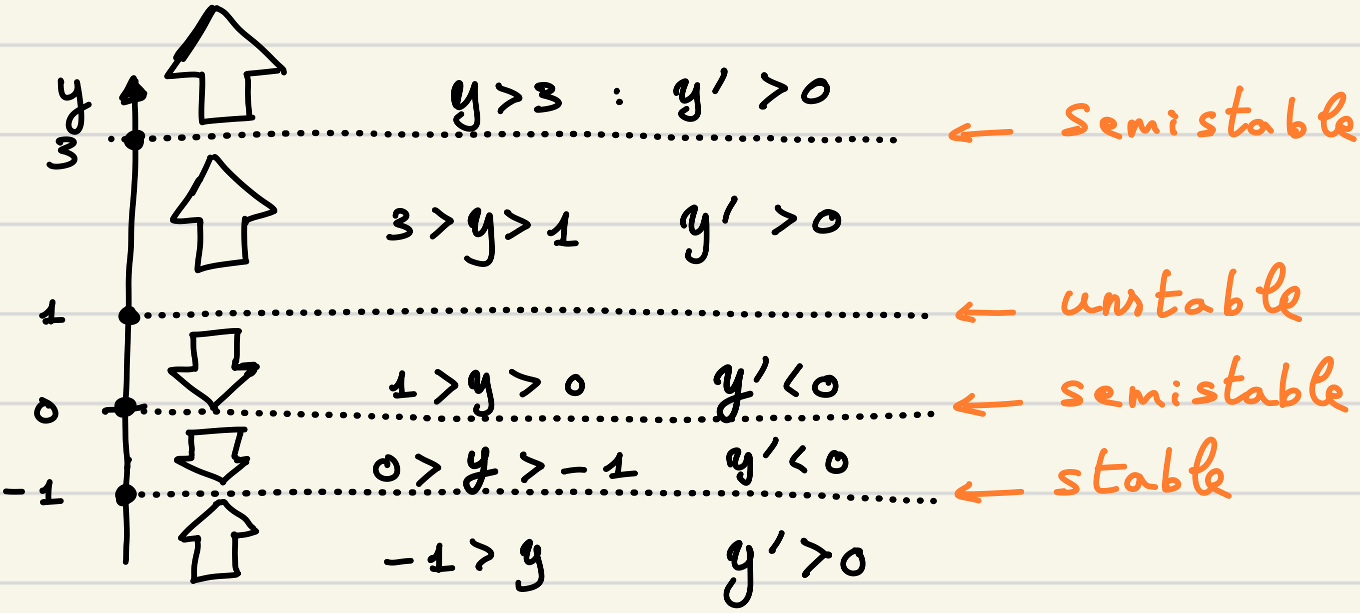
EXAMPLE Find the equilibria, and their properties, for the following DE:

$$\frac{dy}{dx} = \overbrace{y^2(y-1)(y+1)(y^4+1)(y-3)^2}^{h(y)}$$

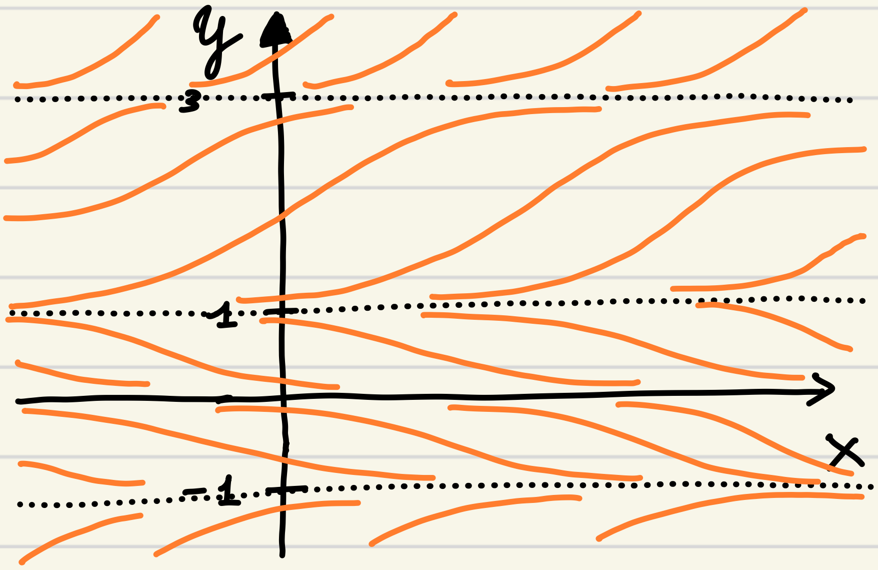
Step 1) Find out for which values of y $h(y)=0$. The polynomial is already fully factored over the real numbers so we can read off its roots:

$$y=0, \quad y=1, \quad y=-1, \quad y=3$$

Now we can plug in values of y in between the equilibria to find the sign of $h(y)$. Note that y^2, (y^4+1), and $(y-3)^2$ are always positive so they don't affect the sign at all! The sign is thus completely determined by $(y+1)(y-1)$



The solutions would thus look like



BIFURCATIONS

Consider the following harvesting model

$$\frac{dP}{dt} = \underbrace{aP - bP^2}_{\text{logistic model}} - \underbrace{h}_{\text{harvesting term}} \qquad \text{where } a, b, h > 0$$

which models a population that grows according to a logistic model but where, at a constant rate, entities are taken away. Examples are fish in a pond with fishers, cockroaches in the math building that leave the basement to find luck on the floors above, flees on the back of a chimp, ...

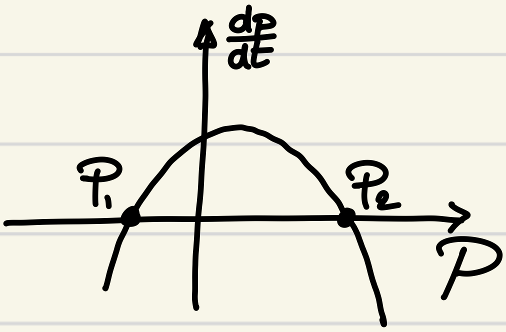
Depending on the values of a, b, and h this model can have 0, 1, or 2 equilibria. If we assume that a, b are fixed and h can be varied then we can plot the qualitative behavior of the model as a function of h.

The roots of $-bP^2+aP-h$ are

$$P_1 = \frac{a+\sqrt{a^2-4bh}}{2b}$$

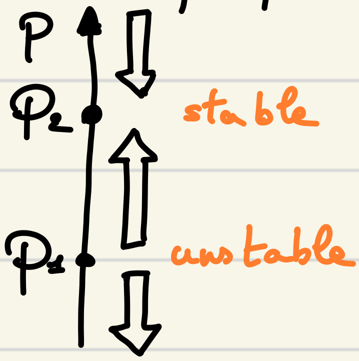
$$P_2 = \frac{a-\sqrt{a^2-4bh}}{2b}$$

Since $b>0$ we have that $\underbrace{-b}_{\text{negative}}P^2+aP-h$ ⇒ sad parabola ☹ looks like



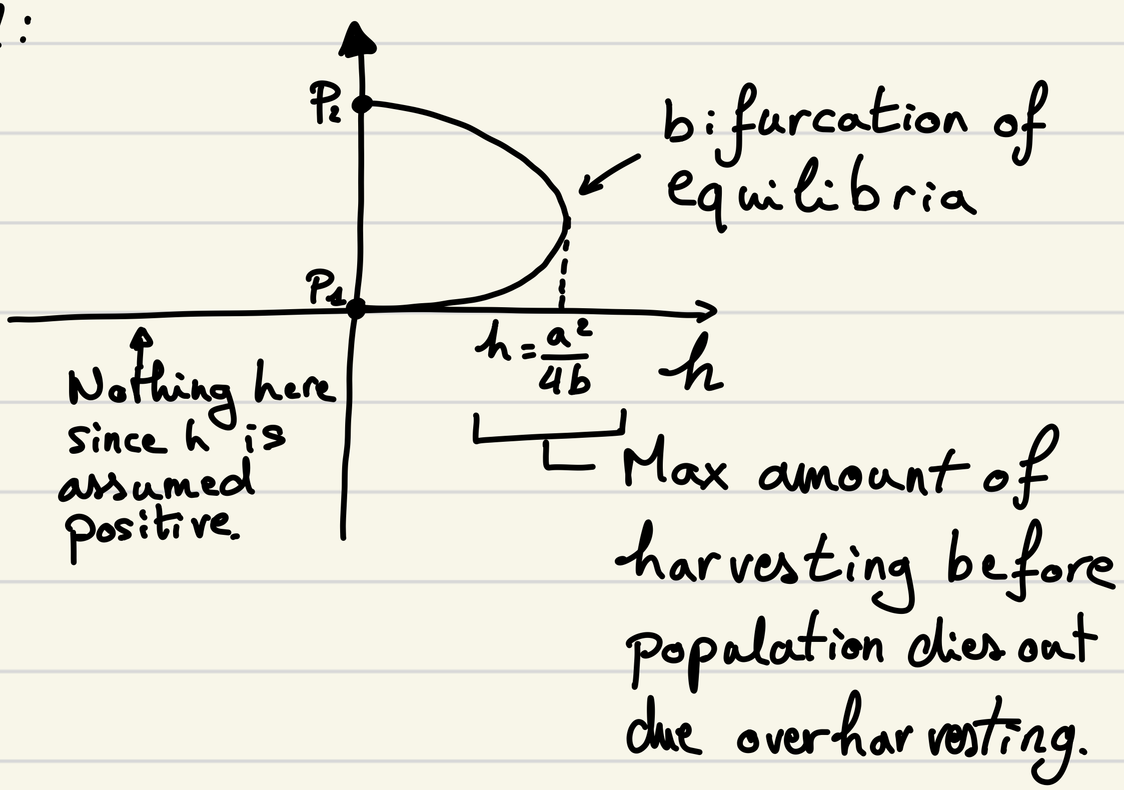
So for $P_1 < P < P_2$ $\quad \frac{dP}{dt} > 0$

while for $P < P_1$ or $P > P_2$ $\quad \frac{dP}{dt} < 0$



Depending on h, we might have two roots, one root with multiplicity 2, or no roots whatsoever. Indeed, if $a^2 - 4bh = 0$, our equilibria merge into a semistable equilibrium. if $a^2 - 4bh < 0$, i.e., $h > \frac{a^2}{4b}$, then there is no equilibrium and $\frac{dP}{dt} < 0$ for all P.

If we plot the equilibria as a function of h we get, what is called, a bifurcation diagram:



SEC 2.3. ACCELERATION-VELOCITY MODELS

By deffinition $\overset{\text{acceleration}}{a} = \frac{d\overset{\text{velocity}}{v}}{dt} = \frac{d^2x}{dt^2}$

For an object with constant mass:

$$F = ma = m\frac{dv}{dt} = m\frac{d^2x}{dt^2}$$

If an object is close to the surface of the earth then it experiences a force, due to gravity, of the form $F_g = mg$.

If this is the only force acting on the object then

$$\cancel{m}\frac{d^2x}{dt^2} = \cancel{m}g$$

so

$$x = \frac{g}{2}t^2 + v_0 t + x_0,$$

which is independent of the mass of the object. In reality not all objects experience the same falling acceleration due to air resistance.

For a generic object, it is not possible to determine the air resistance theoretically. The two most common methods for determining air resistance are

(a) By experiment

(b) Using a computer

For slow moving objects the air resistance can be approximated as

(1) $F_R = -kV \qquad k>0$

or (2) $F_R = -kV^2 \qquad k>0$

In case (1) we find that

$$m\frac{dv}{dt} = mg - kV$$

or $\frac{dv}{dt} + VP = g$ where $P = \frac{k}{m}$

This is a linear DE with $\rho(t) = e^{Pt}$ and thus

$$\frac{d}{dt}\left(ve^{Pt}\right) = ge^{Pt}$$

$$\Rightarrow ve^{Pt} = \frac{g}{P}e^{Pt} + C$$

$$\Rightarrow v = \frac{g}{P} + Ce^{-Pt}$$

If $v = v_0$ at $t=0$, then

$$v_0 = \frac{g}{P} + C$$

so $c = V_0 - \frac{g}{p}$ and

$$\boxed{v = \frac{g}{p} + \left(V_0 - \frac{g}{p}\right) e^{-pt}}$$

If $F_R = -kv^2$, then

$$m\frac{dv}{dt} = mg - kv^2$$

so

$$\frac{dv}{dt} = g - pv^2 \qquad p = \frac{k}{m}$$

which is a seperable DE. In particular

$$\int \frac{dv}{g-pv^2} = \int dt$$

$$\ulcorner \frac{1}{g-pv^2} = \frac{1}{(\sqrt{g}-\sqrt{p}v)(\sqrt{g}+\sqrt{p}v)} = \frac{A}{\sqrt{g}-\sqrt{p}v} + \frac{B}{\sqrt{g}+\sqrt{p}v}$$

$$= \frac{\sqrt{g}A + \sqrt{p}Av + B\sqrt{g} - \sqrt{p}Bv}{g^2 - pv^2}$$

$$= \frac{v\sqrt{p}(A-B) + \sqrt{g}(A+B)}{g^2 - pv^2}$$

$$\Rightarrow \begin{cases} A - B = 0 \\ \sqrt{g}(A+B) = 1 \end{cases} \Longrightarrow \quad A = B = \frac{1}{2\sqrt{g}}$$

$$\llcorner$$

$$\Rightarrow \frac{1}{2\sqrt{g}}\left(\int \frac{dv}{\sqrt{g}-\sqrt{p}v} + \int \frac{dv}{\sqrt{g}+\sqrt{p}v}\right) = t + c_1$$

$$\Rightarrow \frac{1}{2\sqrt{gp}} \ln \left| \frac{\sqrt{g}+\sqrt{p}\,V}{\sqrt{g}-\sqrt{p}\,V} \right| = t + c_1$$

$$\Rightarrow \frac{\sqrt{g}+\sqrt{p}\,V}{\sqrt{g}-\sqrt{p}\,V} = c_2\, e^{2\sqrt{pg}\,t}$$ where $c_2 \neq 0$

If at $t=0$, $v = V_0$ then

$$c_2 = \frac{\sqrt{g}+\sqrt{p}\,V_0}{\sqrt{g}-\sqrt{p}\,V_0}.$$

We will not substitute this now since the equations would become too uggly

$$\Rightarrow \sqrt{g}+\sqrt{p}\,v = \left(\sqrt{g}-\sqrt{p}\,v\right) c_2\, e^{2\sqrt{pg}\,t}$$

$$\Rightarrow v\sqrt{p}\left(1 + c_2 e^{2\sqrt{pg}\,t}\right) = \sqrt{g}\left(c_2\, e^{2\sqrt{pg}\,t} - 1\right)$$

$$\Rightarrow \boxed{v = \sqrt{\frac{g}{p}}\ \frac{c_2\, e^{2\sqrt{pg}\,t} - 1}{c_2\, e^{2\sqrt{pg}\,t} + 1}}$$

SEC 2.4. EULER'S METHOD

Almost no DE of the form $y' = f(x,y)$ is solvable. For example, none of the following DEs are solvable in terms of elementary functions:

$$y' = \sin\left(\frac{x}{y}\right)$$
$$y' = x^2 + \cos(y)$$

To find out what solutions to these DEs look like, Euler invented a method to approximate these. It goes as follows:

① Choose an initial condition $y(x_0) = y_0$, a number of steps, n, and a stepsize h.

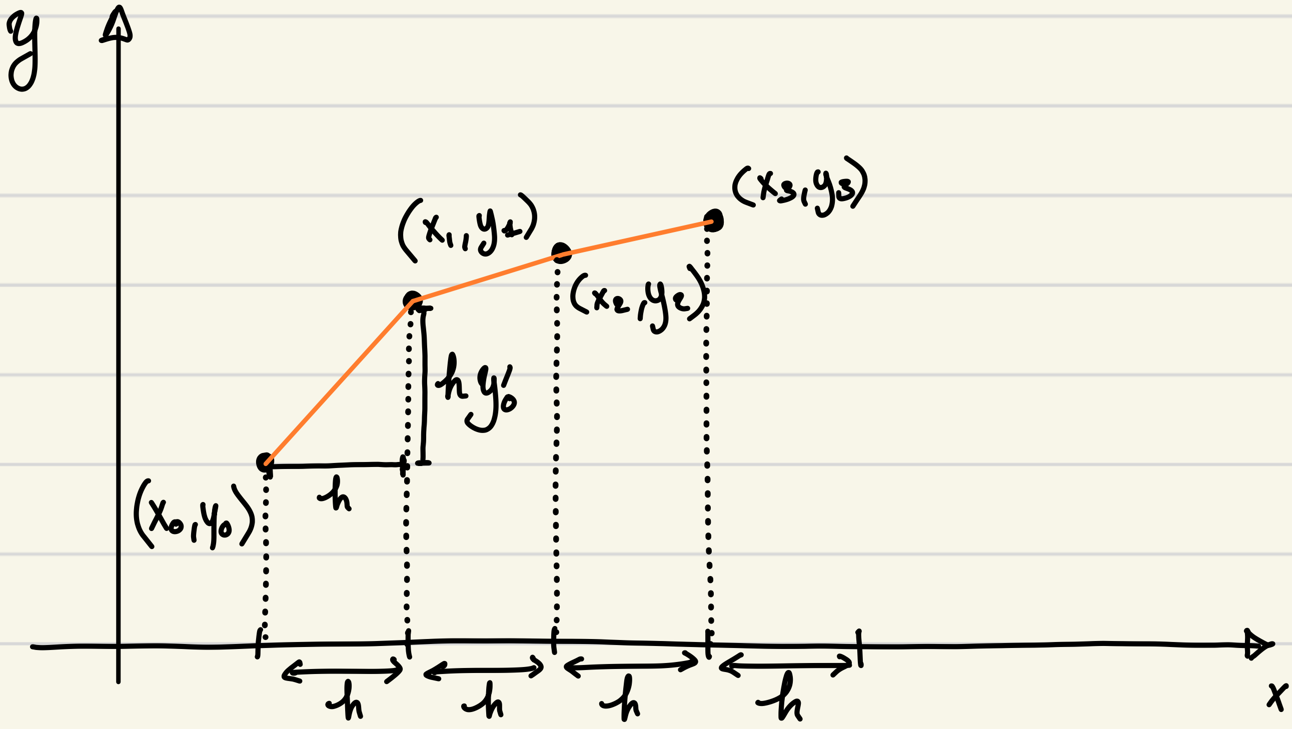
② For $x = x_0$, $y = y_0$, calculate the value of the slope $y_0' = f(x_0, y_0)$

③ Update x_0 to $x_1 = x_0 + h$ and approximate y_1 as follows

$$y_1 = y(x_0 + h) \approx y_0 + h y_0'$$

④ Return to step ② but with x_1, y_1 rather than x_0, y_1 and keep doing this until you've calculated the values of x_n, y_n.

Graphically this procedure looks as follows:



EXAMPLE MIDTERM 1 SPRING 2025

Use Euler's method with $h = 1$ and $(x_0, y_0) = (0,0)$ to approximate the value of $y(3)$ for the following DE

$$y' = y + \sin\left(x \frac{\pi}{4}\right)$$

Solution: We set up a table of values of x_i, y_i, y_i': (see next page)

step	x_i	y_i	$y'_i \; (= y + \sin(x \frac{\pi}{4}))$
0	0	0	0
1	1	$0 + 1 \cdot 0 = 0$	$\frac{\sqrt{2}}{2}$
2	2	$0 + 1 \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2} + 1$
3	3	$\frac{\sqrt{2}}{2} + 1 \cdot \left(\frac{\sqrt{2}}{2} + 1\right) = \sqrt{2} + 1$	

so $\boxed{y(3) \approx \sqrt{2} + 1}$

Sometimes, rather than specifying a number of steps n and a step size h, one specifies

- A number of steps n and an interval $[x_0, x_n]$.
  In this case $h = \frac{x_n - x_0}{n}$.
- A step size h and an interval $[x_0, x_n]$ whose length is a multiple of h. In that case $n = \frac{x_n - x_0}{h}$

CHAPTER 3: LINEAR EQUATIONS OF HIGHER ORDER

First 2 chapters: most DE's were of 1st order.
In this chapter we will investigate higher order DEs of a special kind:

Linear n^{th} order DEs

or LnDE for short.
We will start by considering 2nd order linear DEs (L2DEs) and then generalize to LnDEs.

SEC 3.1. SECOND ORDER LINEAR EQUATIONS

DEFINITION A second order linear differential equation (or L2DE) is of the form

$$A(x)y'' + B(x)y' + C(x)y = F(x)$$

For our purposes we will assume that A, B, C, F are continuous on some interval of interest.

EXAMPLES

(1) $\sin(x)\, y'' + \frac{\ln(x)}{\sqrt{x}} y' + (1+\sqrt{x})y = \tan^{-1}(x)$

is a L2DE where we only care about $x>0$.

(2) $$y''=0$$
is a L2DE with $A(x)=1$, $B(x)=C(x)=F(x)=0$.

(3) $$y''+y^2=0$$
is NOT a L2DE because of the y^2

(4) $$y\,y''+y=1$$
is NOT a L2DE because of $y \cdot y''$

(5) $$y\,y''+y=0 \quad \text{with } y \neq 0$$
is an L2DE for $y \neq 0$.

For L2DEs there exists a much stronger version of the existence & uniqueness theorem than for generic 2DEs

THEOREM (Existence & uniqueness for L2DEs)
If A, B, C, F are continuous on some interval I that contains the value a, and $A(x) \neq 0$ for all x in I then

$$\begin{cases} A(x)y'' + B(x)y' + C(x)y = F(x) \\ y(a) = b_1 \\ y'(a) = b_2 \end{cases}$$

has a unique solution on I.

General L2DEs can be quite hard to solve so we will restrict ourselves to techniques for solving L2DEs where A, B, C are constant, i.e. equations of the form

$$\circledast \qquad ay'' + by' + cy = F(x), \quad a,b,c \in \mathbb{R} \text{ and } a \neq 0.$$

In order to solve $\circledast$ it pays off to look at a special case first

DEFINITION A L2DE is called **homogeneous** if $F(x)=0$.

Homogeneous equations are interesting for 2 reasons

1) In order to solve a general L2DE one must solve an associated homogeneous L2DE first. (This one is obtained by setting $F(x)=0$)

2) Homogeneous L2DEs have the following properties

(i) If y_1, y_2 are solutions, then any function

$$c_1 y_1 + c_2 y_2$$

with c_1, c_2 any constants, is also a solution.

(ii) If y_1, y_2 are linearly independent (i.e. there exists no constant c s.t. $y_1 = c y_2$) then all solutions are of the form

$$c_1 y_1 + c_2 y_2$$

EXAMPLE Say you want to solve

$$y'' + y = 0$$

and you are given that both $y_1(x) = \cos(x)$ and $y_2(x) = \sin(x)$ solve this equation. Since $\sin(x)$ and $\cos(x)$ are linearly independent, all solutions to this equations are of the form $c_1 \cos(x) + c_2 \sin(x)$

But how do we find linearly independent solutions to

$$ay'' + by' + cy = 0 \text{ ?}$$

Answer: try a solution of the form $y = e^{rx}$.

Substituting this solution into the eqn gives:

$$ar^2 e^{rx} + br e^{rx} + ce^{rx} = 0$$

and since $e^{rx} \neq 0$ we get

$$\underbrace{ar^2 + br + c = 0.}_{\text{characteristic equation}}$$

(CE) of the L2DE.

There are 3 possible scenarios

1. The CE has 2 different real solutions: r_1, r_2

In this case $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are LI solutions to the DE. The general solution is then

$$\boxed{y = c_1 e^{r_1 x} + c_2 e^{r_2 x}}$$

2. The CE has only 1 real solution: r_1

In this case $y_1 = e^{r_1 x}$ is a solution but we're missing a second LI solution. To find a second solution, we can use the method of reduction of order: set $y_2(x) = v(x)y_1(x)$ (or shorter $y_2 = vy_1$)

If we plug $y_2 = vy_1$ into the DE we get

$$a(vy_1)'' + b(vy_1)' + cvy_1 = 0$$

$$\Rightarrow av''y_1 + 2av'y_1' + avy_1'' + bv'y_1 + bvy_1' + cvy_1 = 0$$

since $ay_1'' + by_1' + cy_1 = 0$, $ay_1'' = -by_1' - cy_1$

$$\Rightarrow av''y_1 + 2av'y_1' + v(\cancel{-by_1'} - \cancel{cy_1}) + \cancel{bvy_1'} + bv'y_1 + \cancel{cvy_1} = 0$$

$$\Rightarrow av''y_1 + v'(2ay_1' + y_1) = 0$$

Since $y_1 = e^{rx}$, $y_1' = re^{rx}$ and $r = -\frac{b}{2a}$ because it's a root with multiplicity 2

$$\Rightarrow av''e^{rx} + v'\left(2a\left(\frac{-b}{2a}\right)e^{rx} + e^{rx}\right) = 0$$

$$\Rightarrow ae^{rx}v'' = 0 \overset{\substack{a \neq 0 \\ e^{rx} \neq 0}}{\Longrightarrow} v'' = 0 \Rightarrow v = \tilde{C}_1 x + \tilde{C}_2.$$

We only need 1 other LI solution so we can choose $\tilde{C}_1, \tilde{C}_2$ as we please as long as $\tilde{C}_1 \neq 0$. By choosing $\tilde{C}_1 = 1, \tilde{C}_2 = 0$ we get $y_2 = xe^{rx}$.

The general solution is thus

$$\boxed{y = C_1 e^{rx} + C_2 x e^{rx}}$$

3 ⟩ The CE has 2 complex solutions.

Solution: use $e^{a+bi} = e^a(\cos(b) + i\sin(b))$ and choose complex constants c_1, c_2 such that all the complex parts cancel.

The details are for a later section.

EXAMPLES

(1) Solve $y'' + 3y' - 4y = 0$

The CE is $r^2 + 3r - 4 = 0$

or $(r-1)(r+4) = 0$ so $r_1 = 1$, $r_2 = -4$

and

$$y = c_1 e^{x} + c_2 e^{-4x}$$

(2) Solve $y'' + 2y' + y = 0$

The CE is $(r+1)^2 = 0 \Rightarrow r_1 = -1$ as our only solution

Therefore

$$\boxed{y = c_1 e^{-x} + c_2 x e^{-x}}$$

SEC 3.2. GENERAL SOLUTIONS TO LINEAR EQUATIONS

DEFINITION An n'th order linear DE is of the form

$$P_n(x)y^{(n)}+P_{n-1}(x)y^{(n-1)}+\dots+P_1(x)y'+P_0(x)y=F(x)$$

If $P_n(x)\neq 0$ this equation can also be written as

$$\circledast \qquad y^{(n)}+P_{n-1}(x)y^{(n-1)}+\dots+P_1(x)y'+P_0(x)y=f(x).$$

For an equation given by $\circledast$ we have the following strong theorem

THEOREM If $P_1,\dots,P_n$ are continuous on some interval I and $a\in I$ then the following IVP

$$\begin{cases} y^{(n)}+P_{n-1}y^{(n-1)}+\dots+P_1y'+P_0y=0 \\ y(a)=b_0,\ y'(a)=b_1,\dots,y^{(n-1)}(a)=b_{n-1} \end{cases}$$

has a unique solution on I.

Just like for L2DE's we have the following definition.

DEFINITION A homogeneous LnDE is of the form

$$y^{(n)} + P_{n-1}y^{(n-1)} + \dots + P_0 y = 0$$

Note: We won't use $P_i(x)$ anymore but rather P_i for brevity. We will also use the following notation from now on

$$\sum_{i=0}^{n} \text{something}(i) = \text{something}(0) + \text{something}(1) + \dots + \text{something}(n)$$

So an LnDE has the following form:

$$\sum_{i=0}^{n} P_i y^{(i)} = F(x)$$

or (if $P_0(x) \neq 0$)

$$y^{(n)} + \sum_{i=1}^{n} P_i y^{(i)} = f(x).$$

Learning to work with the summatation symbol Σ is essential for any further math classes so it pays of to use it whenever you can!

Just like for HL2DEs we also have a very strong theorem about the solutions of an HLnDE

THEOREM

(i) If $y_1, \dots, y_n$ are solutions to a HLnDE then

$$c_1 y_1 + \dots + c_n y_n \qquad ⊛$$

also solves that DE.

(ii) If $y_1, \dots, y_n$ are LI solutions to a HLnDE then all solutions to that HLnDE are of the form ⊛.

For part (ii) of this theorem we need to know what it means for n functions $y_1, \dots, y_n$ to be LI.

DEFINITION The functions $y_1, \dots, y_n$ are linearly independent (LI) on an interval I if the following equation

$$c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x) = 0 \quad \text{for all } x \in I \qquad (**)$$

in the unknowns $c_1, \dots, c_n$ has only 1 solution, namely $c_1 = c_2 = \dots = c_n = 0$.

The intuition behind this definition is the following. Suppose there was another solution to $(**)$, say with $c_1 \neq 0$. Then we can divide both sides of the equation by c_1 and write

$$y_1 = -\frac{c_2}{c_1} y_2 - \dots - \frac{c_n}{c_1} y_n ,$$

i.e. we can write y_1 as a linear combination of $y_2, \dots, y_n$: it is linearly dependent on the other y's.

But how does one check whether $y_1, \dots, y_n$ are linearly dependent? For this we can use the Wronskian

$$W(y_1, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & \ddots & \vdots \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$$

The following theorem allows us to test for linear independence.

THEOREM (LINEAR INDEPENDENCE)

The functions $y_1, \dots, y_n$ are linearly independent on an interval I if and only if

$$W(y_1, \dots, y_n) \neq 0 \quad \text{on } I$$

EXAMPLE Show that $y_1 = x$, $y_2 = x^2$, $y_3 = x\ln(x)$ are linearly independent for all $x > 0$.

$$W(y_1, y_2, y_3) = \begin{vmatrix} x & x^2 & x\ln(x) \\ 1 & 2x & \ln(x)+1 \\ 0 & 2 & \frac{1}{x} \end{vmatrix} = -2 \begin{vmatrix} x & x\ln(x) \\ 1 & \ln(x)+1 \end{vmatrix} + \frac{1}{x} \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$$

$$= -2\left(x\ln(x) + x - x\ln(x)\right) + \frac{1}{x}\left(2x^2 - x^2\right)$$

$$= -2x + 2x - x$$

$$= -x \neq 0 \quad \text{for } x > 0$$

We now turn to the problem of solving non homogeneous LnDEs.

The following theorem tells us why HLnDEs are worth studying, even when we want to solve general LnDEs.

THEOREM All solutions of a LnDE

$$\sum_{i=0}^{n} P_i(x)\, y^{(i)} = F(x)$$

are of the form

$$y = \underbrace{y_H}_{\substack{\text{all solutions to the associated} \\ \text{HLnDE: } \sum_{i=0}^{n} P_i(x)\, y^{(n-i)} = 0}} + \underbrace{y_P}_{\text{any solution to}}$$

SEC 3.3. HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

If we have an LnDE

$$\sum_{i=0}^{n} P_i(x) y^{(i)} = F(x)$$

then its solutions are given by

$$y = \underbrace{y_H}_{\text{General sol to associated hom equation } = \sum_{i=1}^{n} c_i y_i} + \underbrace{y_P}_{\text{Any solution to the full equation}}$$

If the $P_i(x)$ are nonconstant functions then finding solutions can be hard. If all the P_i are constant, say $P_0 = a_0, P_1 = a_1, \ldots, P_n = a_n$, then there exists a technique to solve such a DE.

For now, we will focus on solving homogeneous LnDEs.

Solving an HLnDE is done the same way as solving an L2DE: we try a solution of the form $y = e^{rx}$.

After substitution of $y = e^{rx}$ in

$$\sum_{i=0}^{n} a_i y^{(i)} = 0$$

we get

$$\underbrace{\sum_{i=0}^{n} a_i r^i = 0}_{\text{Characteristic equation of the DE}} \xrightarrow{\text{In full}} a_0 + a_1 r + a_2 r^2 + \dots + a_n r^n = 0$$

This is a polynomial equation of degree n. The fundamental theorem of algebra tells us that any n^{th} degree polynomial in r can always be written as

$$a_n(\underbrace{r - r_1)(r - r_2) \cdots (r - r_n}_{\text{Roots of the polynomial = solutions to the CE}}) \qquad \circledast$$

Note that some of the roots might be the same and for any complex root, say $r = a + ib$, its conjugate $\bar{r} = a - ib$, is also a root.

DEFINITION The multiplicity of a root of a polynomial equals the number of times that root appears in the factorization $\circledast$

Given a HLnDE with characteristic polynomial

(CP) $a_0 + a_1 r + \dots + a_n r^n = a_n (r - r_1) \cdots (r - r_n)$

there are 4 scenarios

1 All roots are real and have mult 1

In that case: $y_1 = e^{r_1 x}, \dots, y_n = e^{r_n x}$

and thus

$$y = \sum_{i=1}^{n} c_i y_i = \sum_{i=1}^{n} c_i e^{r_i x}$$
$$= c_1 e^{r_1 x} + c_2 e^{r_2 x} + \dots + c_n e^{r_n x}$$

2 All roots are real but some have mult > 1

In this case we won't have enough LI solutions of the form $y_i = e^{r_i x}$. We can apply the method of reduction of order to find the missing solutions. It is beneficial to introduce some new notation first. The HLnDE can also be written as

$$Ly = 0$$

where

$$L = a_0 + a_1 D + a_2 D^2 + \dots + a_n D^n \quad \left(D = \frac{d}{dx}\right)$$
$$= \sum_{i=0}^{n} a_i D^i$$

and by definition

$$Ly = (a_0 + a_1 D + \dots + a_n D^n)\, y$$
$$= a_0 y + a_1 D(y) + \dots + a_n D^n(y)$$

L is called a linear operator. The idea for this name comes from the fact that

$$L(af(x)+bg(x)) = aL(f(x)) + bL(g(x)) \quad a,b: \text{constants}$$

You can multiply linear operators as follows

KL is the operator that sends $y \mapsto K(L(y))$

EXAMPLE If $K = D-a$, and $L = D-b$ then

$$\begin{aligned}(KL)(y) &= K(L(y)) \\ &= K((D-b)(y)) \\ &= K(Dy - by) \\ &= K(Dy) - bK(y) \\ &= D^2y - aDy - bDy + aby\end{aligned}$$

And

$$\begin{aligned}(LK)(y) &= L(K(y)) \\ &= L(Dy - by) \\ &= (D-a)(Dy) - b(D-a)(y) \\ &= D^2y - aDy - bDy + aby \\ &= (KL)(y)\end{aligned}$$

As a matter of fact: any two operators L_1, L_2 of the form

$$L_1 = \sum_{i=0}^{n} a_i D^i \, , \; L_2 = \sum_{i=0}^{n} b_i D^i$$

where all a_i, b_i are constants, satisfy

$$L_1 L_2 = L_2 L_1$$

and their multiplication behaves like polynomial multiplication.

Note: if the a_i, b_i are not constant this doesn't hold anymore.

Back to our problem with roots with multiplicity. Lets start by finding all solutions if the CP can be written as

$$a_n (r - r_1)^n$$

i.e. root r_1 has mult n. In that case $y_1 = e^{r_1 x}$ is a solution but we are missing $n-1$ other solutions. We'll try to obtain these via the method of reduction of order.

Let $y = u y_1$, then since $Ly = 0 \Rightarrow L(u y_1) = 0$

we have that $(a_n D^n + \dots + a_1 D + a_0)(u y_1) = 0$

We know that $a_n r^n + \dots + a_1 r + a_0 = a_n (r - r_1)^n$

So $a_n D^n + \dots + a_1 r + a_0 = a_n (D - r_1)^n$

and our equation becomes

$$\cancel{a_n}(D - r_1)^n (u y_1) = 0$$

$$\Rightarrow (D - r_1)^{n-1}\big((D - r_1)(u y_1)\big) = 0$$

$$\Rightarrow (D - r_1)^{n-1}\left(Du\, y_1 + u D y_1 - r_1 u y_1\right) = 0$$

$$\Rightarrow (D - r_1)^{n-1}\left(D(u) y_1 + \cancel{u r_1} e^{r_1 x} - \cancel{u r_1} e^{r_1 x}\right) = 0$$

$$\Rightarrow (D - r_1)^{n-1}\left(D(u) y_1\right) = 0$$

$$\Rightarrow (D - r_1)^{n-2}\left(D^2(u) y_1\right) = 0$$

$$\vdots$$

$$\Rightarrow D^n(u) y_1 = 0$$

$$\Rightarrow D^n(u) = 0$$

$$\Rightarrow u = c_1 x^{n-1} + c_2 x^{n-2} + \dots + c_{n-1} x + c_n$$

From u we can extract $n-1$ LI solutions:

$$x y_1,\ x^2 y_1,\ x^3 y_1,\ \dots,\ x^n y_1$$

together with y_1 we have n LI solutions so

$$y = c_1 y_1 + c_2 x y_1 + c_3 x^2 y_1 + \dots + c_n x^{n-1} y_1$$
$$= (c_1 + c_2 x + \dots + c_n x^{n-1}) y_1$$

For the more general case we have the following theorem.

THEOREM Each real root r with multiplicity m provides the following terms to the solution

$$(c_1 + c_2 x + \dots + c_m x^{m-1}) e^{rx}.$$

Note that this theorem also works for $m = 1$.

3 Some roots are complex, with mult 1

Lets assume for simplicity that there are only 2 complex roots: $r = a + ib$, and $\bar{r} = a - ib$.

(remember that complex roots always come in pairs)

In that case both e^{rx} and $e^{\bar{r}x}$ solve the DE and together they provide the following linear combination to the solution:

$$c_1 e^{rx} + c_2 e^{\bar{r}x}$$
$$= c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x}$$
$$= e^{ax} \left(c_1 e^{ibx} + c_2 e^{-ibx} \right)$$

Using the fact that $e^{ibx} = \cos(bx) + i \sin(bx)$

$$= e^{ax}\big(c_1 \cos(bx) + c_1 i \sin(bx) +$$
$$c_2 \cos(bx) - c_2 i \sin(bx)\big)$$
$$= e^{ax}\big((c_1+c_2)\cos(bx) + (c_1-c_2) i \sin(bx)\big)$$

Now let $A = c_1 + c_2$ $\left(\text{i.e.} \quad c_1 = \frac{A - iB}{2} \right.$

 $B = i(c_1 - c_2)$ $\left. \phantom{\text{i.e.}} \quad c_2 = \frac{A + iB}{2} \right)$

$$= e^{ax}\big(A \cos(bx) + B \sin(bx)\big)$$

so a pair of conjugate roots provides the following terms to the solution

$$e^{ax}\big(C_1 \cos(bx) + C_2 \sin(bx)\big)$$

4 Some roots are complex with mult > 1. Let r and $\bar{r}$ have multiplicity m. Since the derivative of a complex function in one real variable is calculated the same way as if the complex constants were real numbers, the method of reduction of order tells us that

$$(c_1 + c_2 x + \dots + c_m x^{m-1})e^{rx} \quad \text{and} \quad (c_{m+1} + c_{m+2} x + \dots + c_{2m}^{m-1})e^{\bar{r}x}$$

give us $2m$ LI complex solutions.

By using the same technique as for case 3 we can convert these to 2m LI real solutions of the form

$$(A_1+A_2x+\dots+A_{m-1}x^{m-1})e^{ax}\cos(bx) + (B_1+B_2x+\dots+B_{m-1}x^{m-1})e^{ax}\sin(bx)$$

ALL TOGETHER

THEOREM (CONTRIBUTIONS OF ROOTS)

Let r be a root of the CP of an HLnDE with mult m then

(i) if r is real, it contributes the following term to the general solution

$$(c_1+c_2x+\dots+c_mx^{m-1})e^{rx}$$

(ii) if r is complex, i.e. $r = a+ib$, with conjugate root $\bar{r} = a-ib$, and has multiplicity m, then these two roots contribute the following terms to the general solution.

$$(A_1+A_2x+\dots+A_mx^{m-1})e^{ax}\cos(bx) + (B_1+B_2x+\dots+B_mx^{m-1})e^{ax}\sin(bx)$$

So in practice one solves a HLnDE as follows:

(1) Set up the characteristic polynomial (CP)

(2) Determine the roots of (CP). Write these down in a table together with their multiplicities, and group the pairs of conjugate roots together:

	root	mult	
Real roots	r_1	m_1	
	r_2	m_2	
	$\vdots$		
	r_k	m_k	complex conjugate roots have the same mult
	r_{k+1}	m_k	
	$\vdots$		
Pairs of complex roots.	r_{n-1}	m_{n-1}	complex conjugate roots have the same mult
	r_n	m_{n-1}	

(3) Determine the LI solutions that each real root and each pair of complex roots provides and add all of these together.

EXAMPLE Solve the following DE

$$(D-1)(D+2)^2(D^2+2)(D^6+1)y=0$$

Solution (1) The characteristic polynomial is

$$(r-1)(r+2)^2(r^2+2)(r^2+1)^3$$

which has the following roots :

(2)

	Root	Mult
r_1	1	1
r_2	-2	2
r_3	$0+i\sqrt{2}$	1
r_4	$0-i\sqrt{2}$	1
r_5	$0+i$	3
r_6	$0-i$	3

(3) These give rise to the following solution

$$
\begin{aligned}
y = {} & c_1 e^{x} \\
& +(c_2+c_3x)\,e^{-2x} \\
& +c_4\cos(\sqrt{2}\,x)+c_5\sin(\sqrt{2}\,x) \\
& +(c_6+c_7x+c_8x^2)\cos(x)+(c_9+c_{10}x+c_{11}x^2)\sin(x)
\end{aligned}
$$

REVISION MIDTERM 1

Sections 1.2 - 1.6, 2.1 - 2.4, 3.1 - 3.3

IMPORTANT NOTE:

These slides do not revise all the material!
They're not sufficient to revise for the exam!

They're meant as a guide to some of the key concepts in the course.

CHAPTER 1

Mainly: the different DE’s and techniques to solve them

Sec 1.2. Integrals as general and particular solutions

EQUATION

- The **SIMPLEST** equation $$\frac{d^n y}{dx^n} = f(x)$$

Solution method: integrate both sides n times to obtain

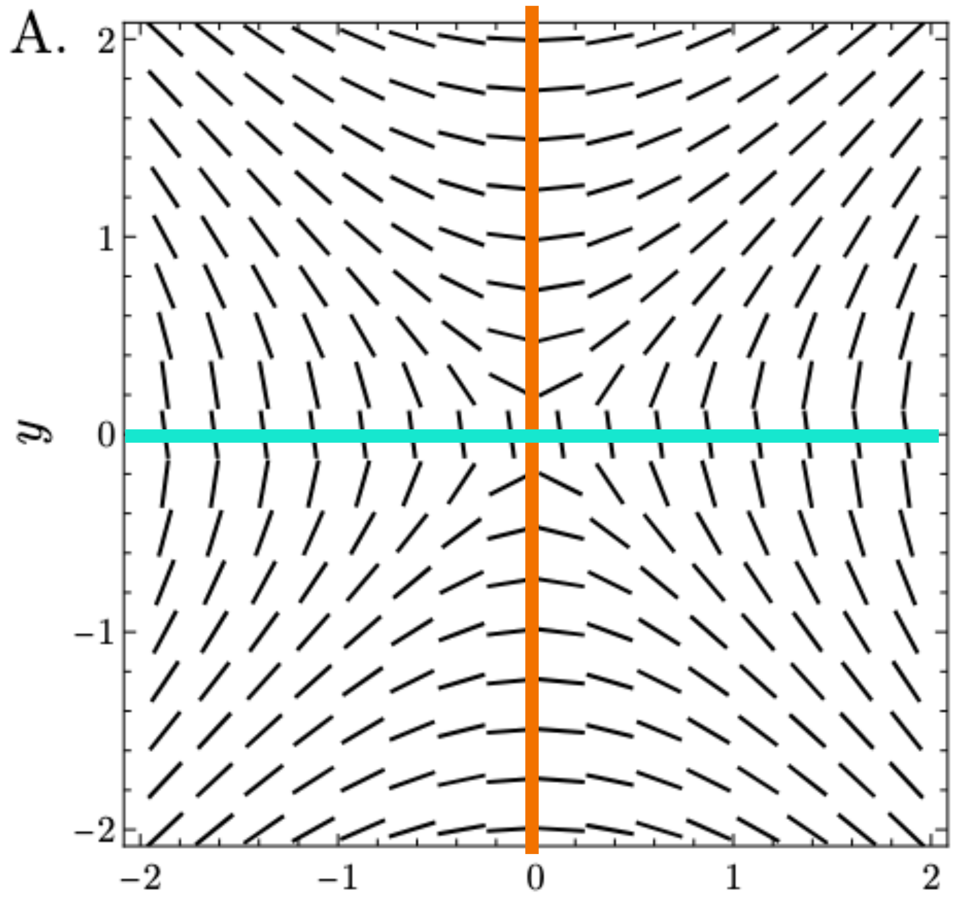
$$y = \int \dots \int f(x)\ dx^n$$

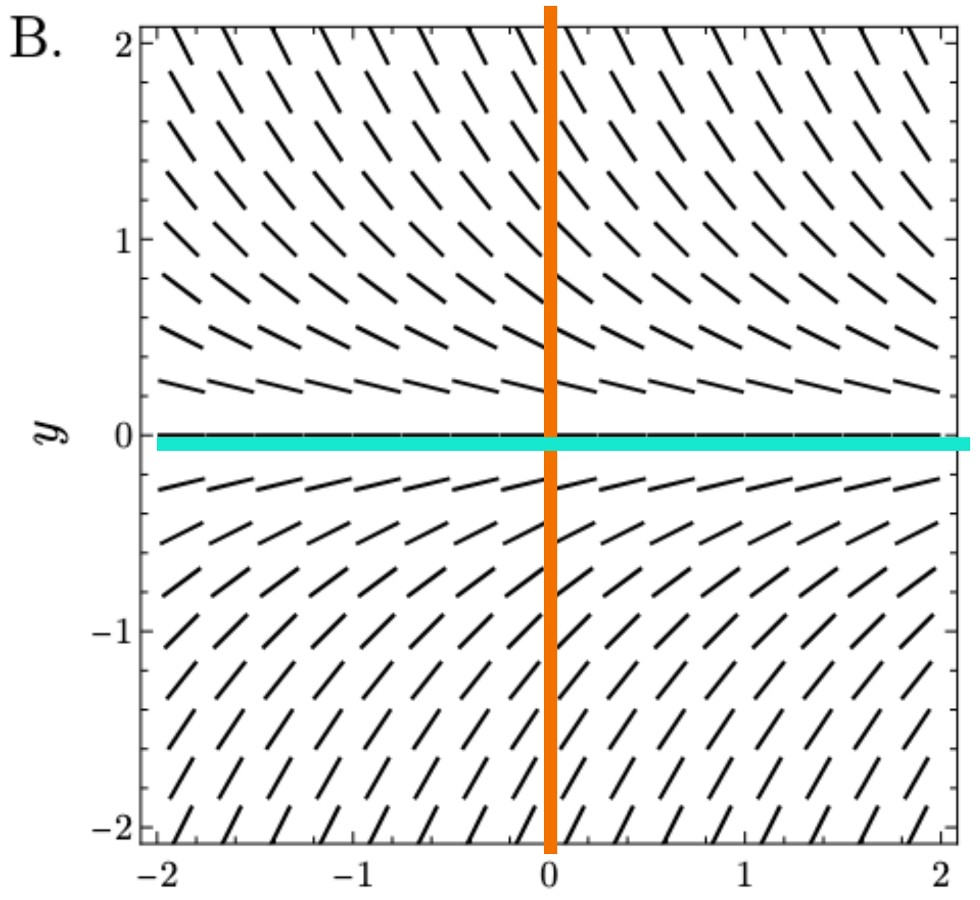
Note: don't forget to integrate the integration constants!

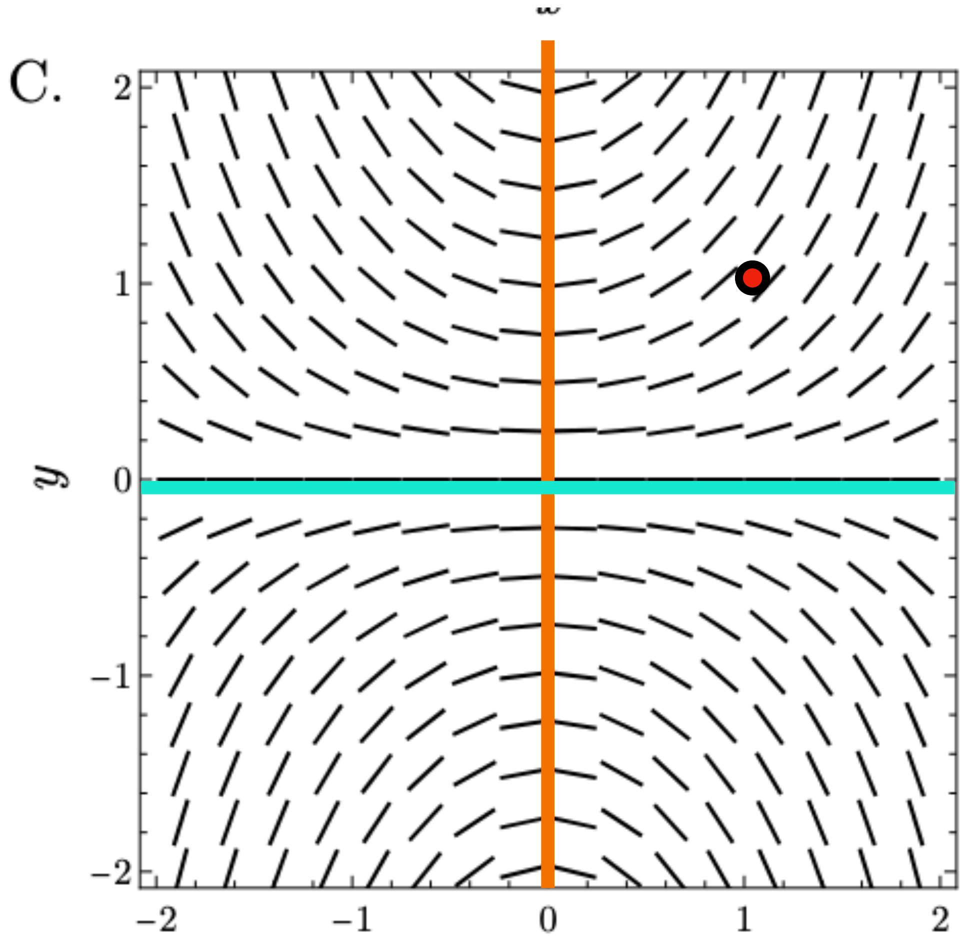
Sec 1.3. Slope fields and solution curves

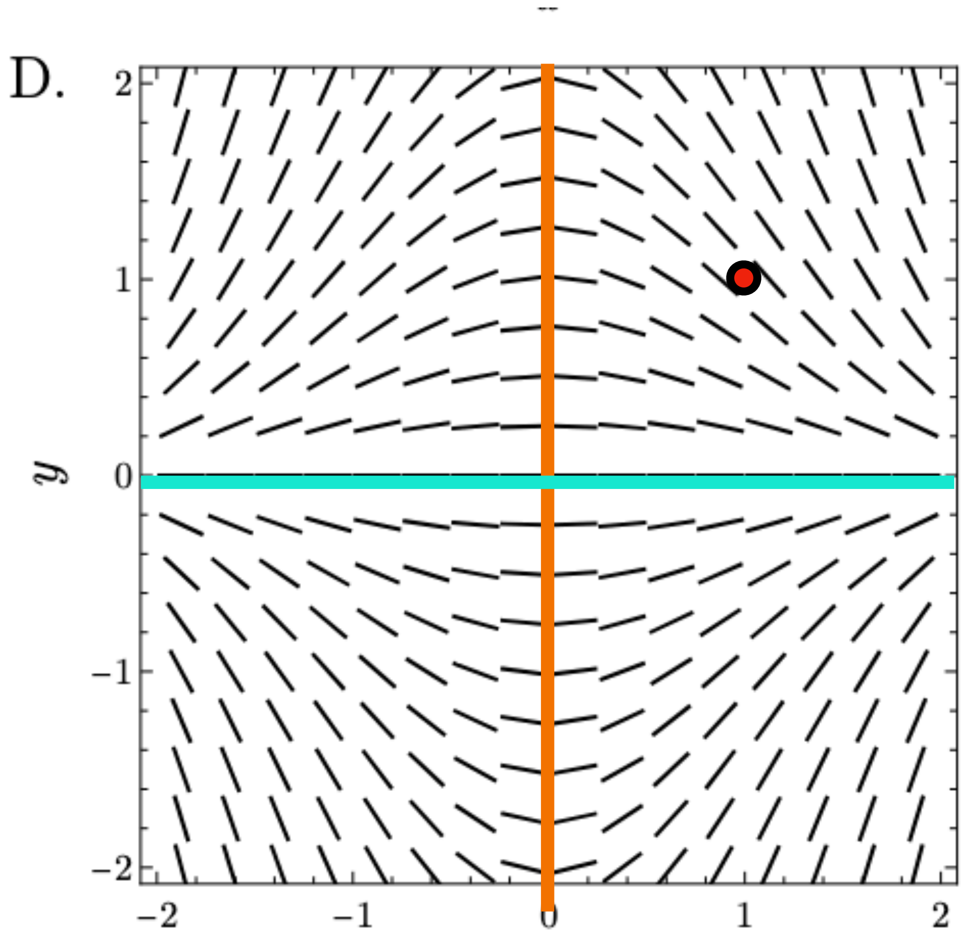
TYPICAL QUESTION (midterm 1 fall 2024)

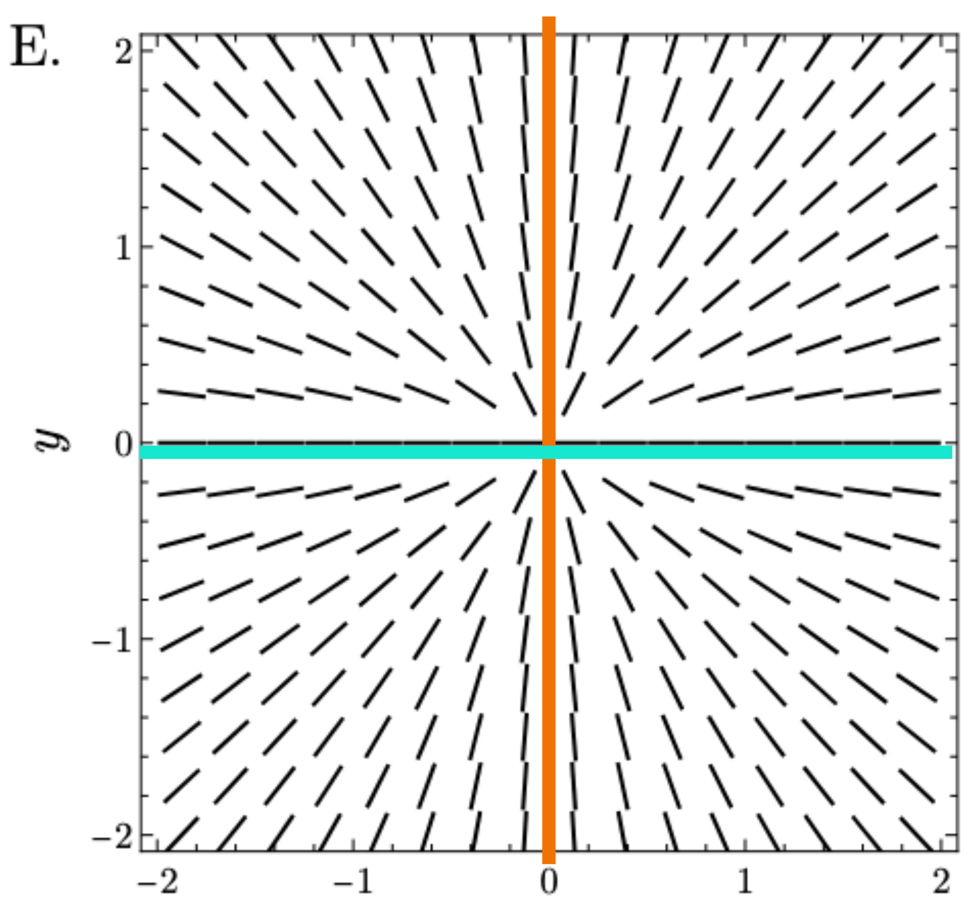
1. Identify the slope field of the differential equation $y' = -xy$.

A.


B.


C.


D.


E.


Solution method: fill in some well chosen points.

If $x = 0$, $y' = 0$ so B and E are impossible. If $y = 0$, $y' = 0$ so A is impossible.

If $x = 1$,$y = 1$, $y' = -1$ so the answer is D

Sec 1.4. Separable equations and applications

EQUATION

- **SEPARABLE** equation $$\frac{dy}{dx} = g(x)h(y)$$

Solution method:

1. Assume $h(y) = 0$. Values of y for which this is true are the singular solutions.
2. Assume $h(y) \neq 0$. Divide both sides by $h(y)$, multiply both sides with dx, and integrate to get
$$\int \frac{dy}{h(y)} = \int g(x)\ dx$$
3. Combine singular and general solutions

Note: you don't need an integration constant for the integral on the left-hand-side

Sec 1.5. Linear first order ODEs

EQUATION

- **LINEAR FIRST ORDER** equation $y' + yP(x) = Q(x)$

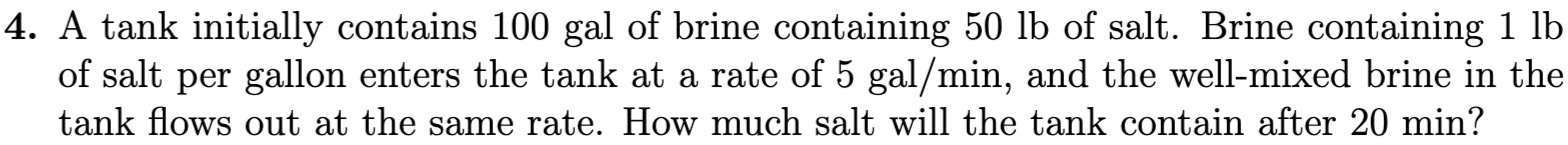
Solution method:

1. Calculate $\rho(x) = e^{\int P(x)dx}$
2. Multiply the DE by $\rho(x)$ to get $y'\rho(x) + yP(x)\rho(x) = Q(x)\rho(x)$
3. Recognise left-hand-side as $\big(y\rho(x)\big)'$
4. Integrate both sides w.r.t. x.
5. Solve for y

Note: you don't need an integration constant for $\displaystyle\int P(x)dx$

Sec 1.5. Linear first order ODEs

TYPICAL QUESTION (Spring 2023 Midterm 1)



4. A tank initially contains 100 gal of brine containing 50 lb of salt. Brine containing 1 lb of salt per gallon enters the tank at a rate of 5 gal/min, and the well-mixed brine in the tank flows out at the same rate. How much salt will the tank contain after 20 min?

Solution method:

1. Set up the DE as described in notes (or book)
2. Solve DE and find c by using initial values (in this case at $t = 0$, $x = 50$)
3. Plug in $t = 20$ in solution

Sec 1.6. Substitution methods

GENERAL METHOD

To substitute $v(x) = f(x, y)$ into a DE you need to express y as a function of $v(x)$ and x, say $y = g(v(x), x)$ and use this function to find $\dfrac{dy}{dx} = \dfrac{d}{dx} g(v(x), x)$. (Don't forget that v is a function of x!)

Then you replace every occurrence of y and $\dfrac{dy}{dx}$ by their expressions in terms of $v(x)$ and x

Note: don't forget to substitute $v(x)$ back in terms of y and x at the end!
You want a solution of the form $y = \ldots$, NOT $v = \ldots$

Sec 1.6. Substitution methods

EQUATIONS

- **HOMOGENEOUS** equation

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Solution method:

1. Set $v(x) = \dfrac{y(x)}{x}$, and thus $y(x) = v(x)x$
2. From this it follows that $\dfrac{dy}{dx} = \dfrac{d}{dx}(v(x)x) = \dfrac{dv}{dx}x + v(x)$
3. Replacing $y(x), y'(x)$ in the DE gives $\dfrac{dv}{dx}x + v = f(v)$, which is separable

Sec 1.6. Substitution methods

EQUATIONS

- **Bernouilli** equation $$\frac{dy}{dx} + yP(x) = Q(x)y^n$$

Solution method:

1. Set $v(x) = y(x)^{1-n}$, and thus $y(x) = v(x)^{\frac{1}{1-n}}$
2. From this it follows that $\frac{dy}{dx} = \frac{d}{dx}(v(x)^{\frac{1}{1-n}}) = \frac{1}{1-n}v(x)^{\frac{n}{1-n}}\frac{dv}{dx}$
3. Replacing $y(x), y'(x)$ in the DE then gives a linear first order equation

Sec 1.6. Substitution methods

Note: typically the Bernouilli equation comes with the demand that $y \neq 0$ *or* $y > 0$*. This is purely so you won't have to worry about taking roots of negative numbers or dividing by* 0*. Just write your solution and add that it only works for those x for which* $y \neq 0$ *or* $y > 0$ *(depending on which demand it is).*

You don't have to calculate the values of x *for which this is true!*

Sec 1.6. Substitution methods

EQUATIONS

- **(possibly) EXACT** equation $M(x,y)dx + N(x,y)dy = 0$

Proving exactness on open rectangle R in xy-plane:

If $M, N, \dfrac{\partial M}{\partial y}, \dfrac{\partial N}{\partial x}$ are continuous on R then DE is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Sec 1.6. Substitution methods

EQUATIONS

- **(possibly) EXACT** equation $M(x, y)dx + N(x, y)dy = 0$

Solution method:

1. If DE is exact there exists an $F(x, y)$ such that $F(x, y) = c$ is the solution and
   A. $\dfrac{\partial F(x, y)}{\partial x} = M(x, y),$
   B. $\dfrac{\partial F(x, y)}{\partial y} = N(x, y)$

Sec 1.6. Substitution methods

EQUATIONS

- **(possibly) EXACT** equation $M(x,y)dx + N(x,y)dy = 0$

Solution method (continued):

2. From (A) we see that $F(x,y) = \int M(x,y)dx$

   Note: This is a partial integral, so integration const = $g(y)$.

3. Now plug $F(x,y)$ from step 2. into (B) to obtain a differential equation for $g(y)$
4. Solve this DE and you've found $F(x,y)$, the solution is then $F(x,y) = c$

Sec 1.6. Substitution methods

EQUATIONS

- **REDUCIBLE EQUATIONS**

(A) Doesn't contain x

$$f\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$$

(B) Doesn't contain y

$$f\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$$

Sec 1.6. Substitution methods

EQUATIONS

- **REDUCIBLE TYPE A**

$$f\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$$

Solution method:

1. Use y as variable and substitute $v(y) = \frac{dy}{dx}$, so $\frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = \frac{dv}{dy}v$
2. The new equation becomes $f\left(y, v, \frac{dv}{dy}v\right) = 0$
3. Solve this equation to find v as a function of y: $v = g(y)$
4. Solve $v = \frac{dy}{dx} = g(y)$ for y as a function of x

Sec 1.6. Substitution methods

EQUATIONS

- **REDUCIBLE TYPE B**

$$f\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$$

Solution method:

1. Keep x as variable and substitute $v(x) = \dfrac{dy}{dx}$, so $\dfrac{d^2y}{dx^2} = \dfrac{dv}{dx}$
2. The new equation becomes $f\left(x, v, \dfrac{dv}{dx}\right) = 0$
3. Solve this equation to find v as a function of x: $v = g(x)$
4. Solve $v = \dfrac{dy}{dx} = g(x)$ for y as a function of x

CHAPTER 2

Mainly: applications of the DE’s of chapter 1

Sec 2.1. Population models

EQUATIONS

- **MORE GENERAL POPULATION MODEL** $$\frac{dP}{dt} = aP - bP^2$$

Solution method:

1. This is a separable equation
2. Singular solutions: values of P such that $aP - bP^2 = 0$
3. General solution: need to use partial fraction decomposition to compute integral

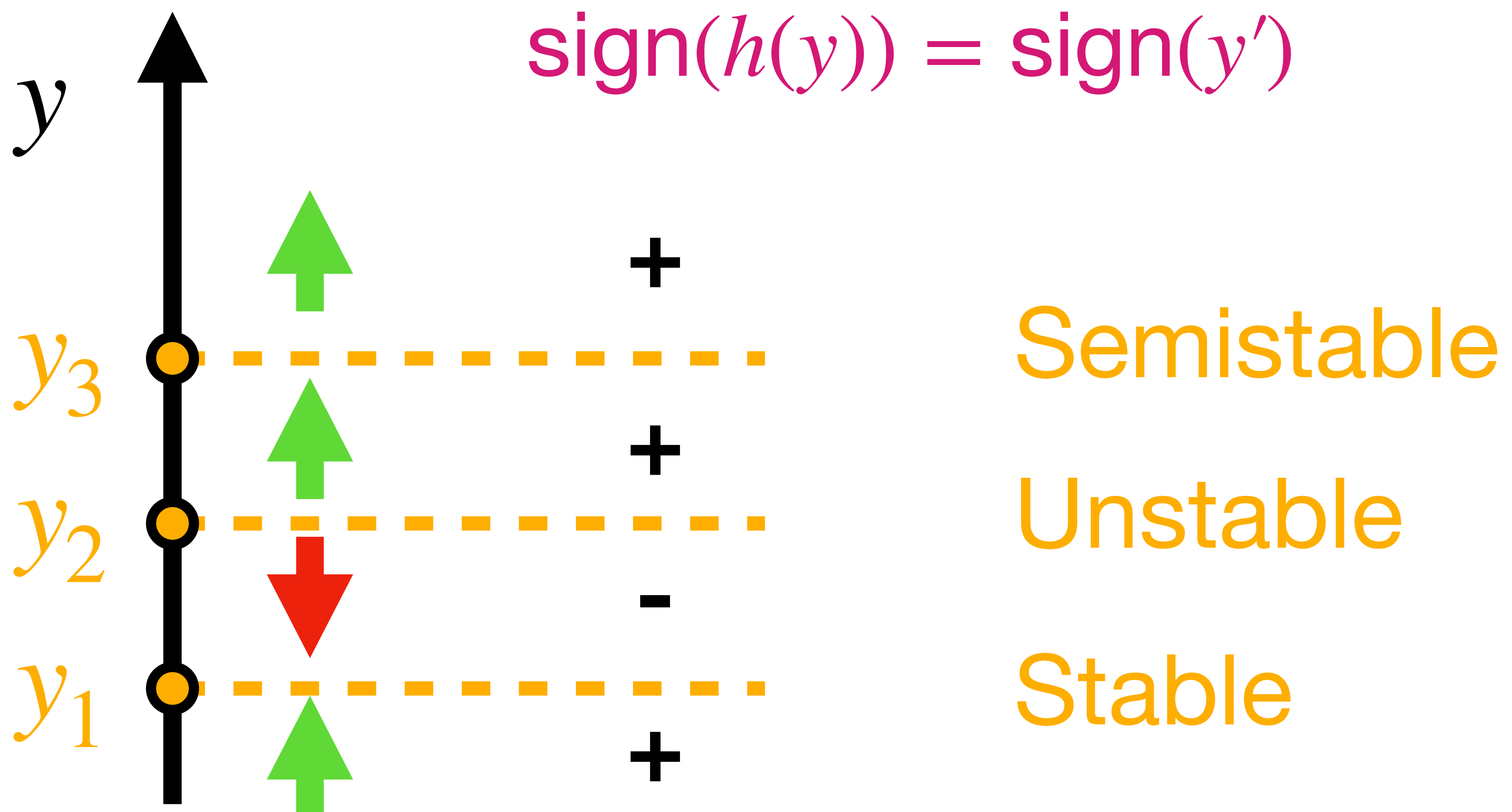
Sec 2.2. Equilibria and bifurcations

EQUATIONS $$\frac{dy}{dx} = h(y) \qquad \left(\text{or} \quad \frac{dP}{dt} = h(P) \right)$$

Equilibria (or critical points): Values of y for which $h(y) = 0$

Types of equilibria: depends on behaviour solutions near equilibria.
Can be determined using sign of $h(y)$ (even **without solving** the DE!)

Example



sign$(h(y))$ = sign(y')

Semistable

Unstable

Stable

Sec 2.2. Equilibria and bifurcations

EQUATIONS

- **POPULATION WITH HARVESTING** $$\frac{dP}{dx} = aP - bP^2 - h$$

# of equilibria depends on value of h

Value of h for which # of equilibria changes = **bifurcation point**

Diagram that shows values of equilibria as function of h = **bifurcation diagram**

Sec 2.3. Acceleration-Velocity models

EQUATIONS

Equations are set up using Newton's second law: $F_{total} = m\frac{dv}{dt}$

Air resistance can be linear $F_R = \pm kv$ or quadratic $F_R = \pm kv^2$ and always opposes direction of motion:

sign depends on how you set up your axes!

Notes:

(1) the equation you obtain is always separable

(2) You don't need to solve the equations to get info on equilibria

Sec 2.4. Eulers method

IDEA

A solution to an IVP of the from $y' = f(x, y)$, $f(x_0) = y_0$ can be approximated with step size h and number of steps n by repeatedly replacing (x_i, y_i) by $(x_{i+1}, y_{i+1}) = (x_i + h, y_i + hf(x_i, y_i))$ until a point (x_n, y_n) is obtained

CHAPTER 3

Mainly: homogeneous linear DEs with constant coefficients

Sec 3.3 Homogeneous equations with constant coefficients

EQUATION

$$\sum_{i=0}^{n} a_i D^i y = 0$$

Solution method:

1. Set up the characteristic equation $\sum_{i=0}^{n} a_i r^i = 0$
2. Solve the equation and obtain roots $r_1, \ldots, r_n$ (some of them possibly equal)
3. Each real root r with multiplicity m provides the following term to the solution $(c_0 + c_1 x + \cdots c_{m-1} x^{m-1}) e^{rx}$
4. Each **pair** of complex roots $r = a + \mathbf{i}b, \bar{r} = a - \mathbf{i}b$ provides the following terms to the solution $(c_0 + c_1 x + \cdots c_{m-1} x^{m-1}) e^{ax} \cos(bx) + (c_m + c_{m+1} x + \cdots c_{2m-1} x^{m-1}) e^{ax} \sin(bx)$

Sec 3.3 Homogeneous equations with constant coefficients

EQUATION

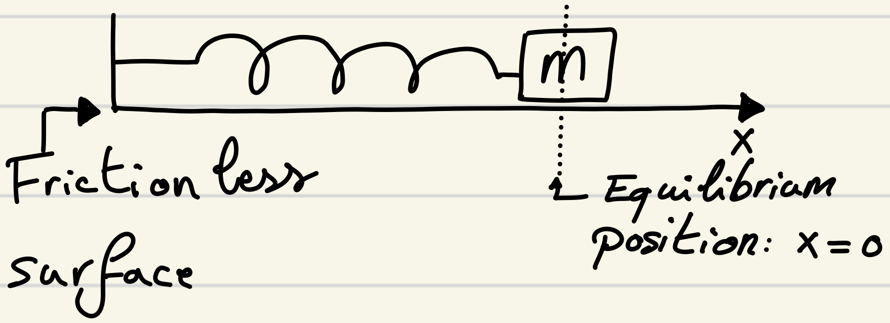
$$\sum_{i=0}^{n} a_i D^i y = 0$$

IMPORTANT NOTES

1. Each constant in each term provided by the various roots is a different constant (I used the same notation in both real and complex case because the correct notation is too cumbersome and distracting)
2. You should get as many constants $c_1, \ldots, c_n$ as the degree of the DE

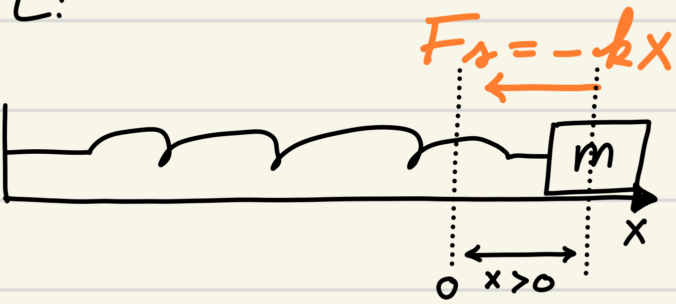
SEC 3.4 MECHANICAL VIBRATIONS

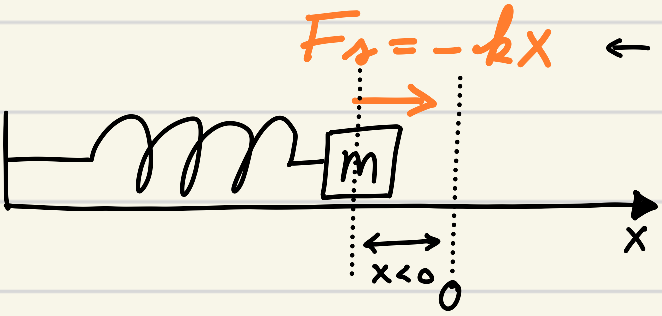
Consider a spring with mass m on a frictionless surface.



If one stretches or compresses the spring then the spring will apply a force opposite to the direction of the displacement:

spring constant >0
Large k ⇒ stiff spring





← Note that x is negative here so F_s is positive.

According to Newton's 2nd law:

$$F = m\frac{d^2x}{dt^2}$$

which, in our case, becomes

$$F_s = -kx = m\frac{d^2x}{dt^2}$$

So $$m\frac{d^2x}{dt^2}+kx=0$$

$$\Rightarrow \frac{d^2x}{dt^2}+\frac{k}{m}x=0$$

Let $\omega_0=\sqrt{\frac{k}{m}}$ then our equation becomes

$$\boxed{\frac{d^2x}{dt^2}+\omega_0^2x=0}$$

This is a HL2DE so we can solve it using the techniques from previous section.

Step 1) The CE is

$$r^2+\omega_0^2=0$$

2) Its solutions are

root	value	mult
r	$0+i\omega_0$	1
$\bar{r}$	$0-i\omega_0$	1

3) The general solution to the equation is therefore

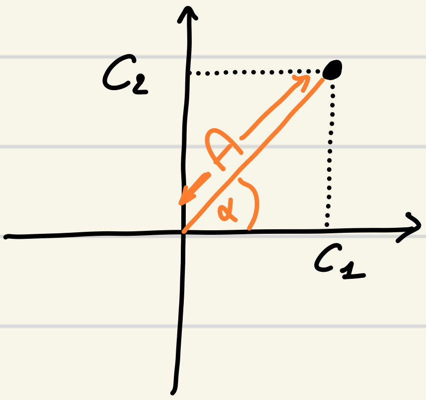
(*) $$x=C_1\cos(\omega_0t)+C_2\sin(\omega_0t)$$

Typically this solution is presented in another form, namely

(**) $$x=A\cos(\omega_0t-\alpha),\quad A>0,\ \alpha\in[0,2\pi[$$

It is not too hard to show that these are equivalent:

In $\circledast$ C_1, C_2 are independent and can take on any value. Specifying C_1, C_2 amounts to choosing a point in a plane with an origin (a 2Dim vector space)



Such a point can also be represented using polar coordinates: $C_1 = A\cos(\alpha)$, $C_2 = A\sin(\alpha)$.

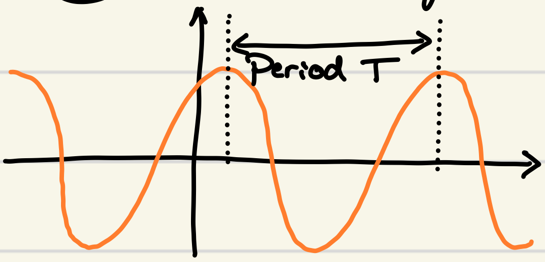
If we plug this form into $\circledast$ we get

$$\begin{aligned} X &= A\cos(\alpha)\cos(\omega_0 t) + A\sin(\alpha)\sin(\omega_0 t) \\ &= A\left(\cos(\alpha)\cos(\omega_0 t) + \sin(\alpha)\sin(\omega_0 t)\right) \\ &= A\cos(\omega_0 t - \alpha) \end{aligned}$$

since

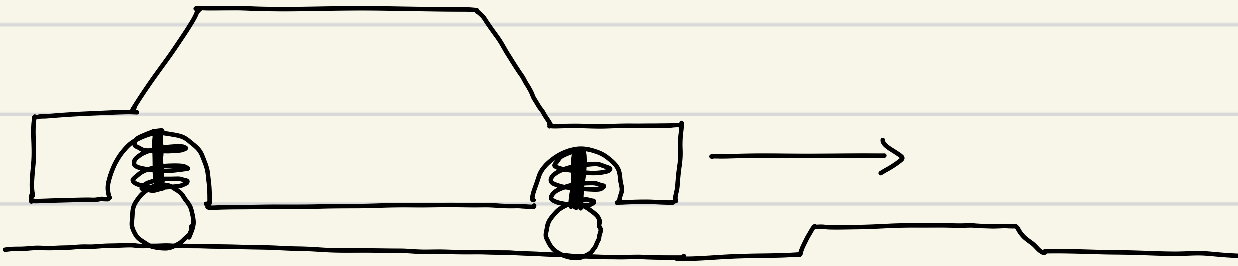
$$\cos(U+V) = \cos(U)\cos(V) - \sin(U)\sin(V)$$

The solution ($**$) has the following graph



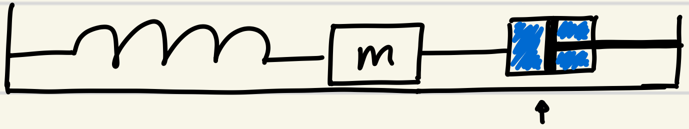
$$T = \frac{2\pi}{\omega_0}$$

Now immagine a car where springs are used to "dampen" shocks as follows:



When the car drives over a bump (so compresses the springs) then you would bounce forever: Big fun but not very safe.

In engineering one often uses a dashpot to avoid this problem



Tube filled with liquid that results in a force, F_R, opposite to velocity

$$F_R = -\underbrace{c}_{\text{dashpot constant}}v$$

In that case, Newton's second law gives

$$-kx - cv = m\frac{d^2x}{dt^2}$$

and since $v = \frac{dx}{dt}$:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

Solution:

1) The CE is $mr^2 + cr + k = 0$

2) Its solutions are :

A. If $c^2 - 4km > 0$

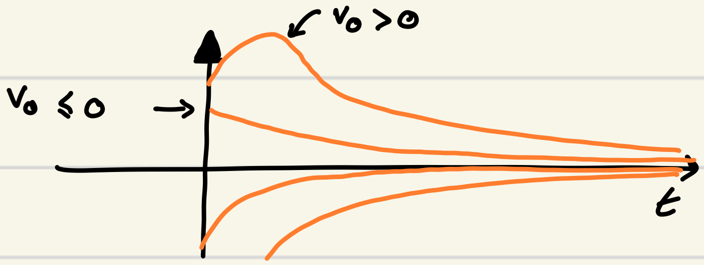
$$r_1 = \frac{-c + \sqrt{c^2 - 4km}}{2m} < 0$$

$$r_2 = \frac{-c - \sqrt{c^2 - 4km}}{2m} < 0$$

and the solution takes the form

$$x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Since $r_1, r_2 < 0$ such solutions look like



Here the dashpot constant c is so big that no oscillations can occur.

This is called the OVERDAMPED case

B. If $c^2 < 4mk$ there are two complex solutions to the CE:

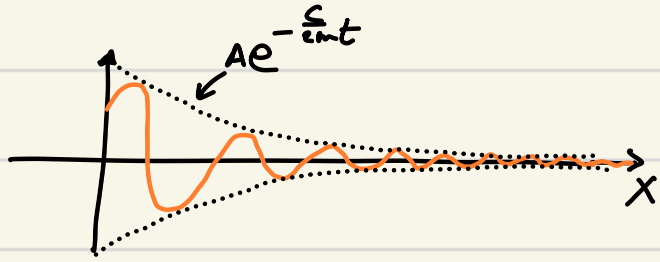
$$r = -\frac{c}{2m} + i\omega_1$$

where $\omega_1 = \sqrt{4km - c^2}$

$$\bar{r} = -\frac{c}{2m} - i\omega_1$$

So
$$x = e^{-\frac{c}{2m}t}\left(C_1 \cos(\omega_1 t) + C_2 \sin(\omega_1 t)\right)$$
$$= A e^{-\frac{c}{2m}t} \cos(\omega_1 t - \alpha)$$

These solutions look like



Because there are still oscillations occurring, this case is called the UNDERDAMPED case.

C. If $c^2 = 4km$ then there is only one root $r = -\frac{c}{2m}$ and the solution becomes

$$x = (c_1 + c_2 t) e^{-\frac{c}{2m} t}$$

Its solutions look very similar to those of case (a). Since this case is the intermediate case between the underdamped & overdamped cases it is called the CRITICALLY DAMPED case.

SEC 3.5 NONHOMOGENEOUS EQUATIONS

A nonhomogeneous LnDE with constant coefficients has the form

$$\underbrace{\left(\sum_{i=0}^{n} a_i D^i\right)}_{L \;\leftarrow\; \text{linear operator}}(y) = F(x) \qquad \circledast$$

and its associated homogeneous equation is

$$L(y) = 0. \qquad \circledast\circledast$$

If y_H solves $\circledast$ and y_p solves $\circledast\circledast$ then, since L is linear,

$$\begin{aligned} L(y_H + y_p) &= L(y_H) + L(y_p) \\ &= 0 + F(x) \end{aligned}$$

so $y_H + y_p$ also solves $\circledast$

It turns out that all solutions to ⊛ are of the form

$$y_H + y_P$$

General solution to ⊛⊛. Has n integration const

Particular solution to ⊛. Has no integration const

But how does one obtain y_P?
For simple cases of $F(x)$ one can use trial and error.

EXAMPLES (1) Find y_P for

$$y'' + 3y' + 4y = 3x + 2$$

Try $y_P = ax + b \Rightarrow y_P' = a, \quad y_P'' = 0$

$$\Rightarrow 3a + 4b + 4ax = 3x + 2$$

$$\Rightarrow \begin{cases} 4a = 3 \\ 3a + 4b = 2 \end{cases} \Rightarrow \begin{cases} a = \frac{3}{4} \\ \frac{9}{4} + 4b = 2 \end{cases} \Rightarrow \begin{cases} a = \frac{3}{4} \\ b = -\frac{1}{16} \end{cases}$$

So $\boxed{y_P = \frac{3}{4}x - \frac{1}{16}}$

(2) $3y'' + y' - 2y = 2\cos(x)$

Neither $y_p = a\cos(x)$, nor $y_p = a\sin(x)$ work.
$y_p = a\cos(x) + b\sin(x)$ does the trick, however.
Let $C = \cos(x)$, $S = \sin(x)$ so $C' = -S$, $S' = C$
and thus

$$y_p = aC + bS, \quad y_p' = -aS + bC, \quad y'' = -aC - bS$$

so $-3aC - 3bS - aS + bC - 2aC - 2bS = 2C$

$\Leftrightarrow (-3a + b - 2a)C + (-3b - a - 2b)S = 2C$

$$\Rightarrow \begin{cases} -5a + b = 2 \\ a + 5b = 0 \end{cases} \Rightarrow \begin{cases} 26\,b = 2 \\ 26\,a = -10 \end{cases} \Rightarrow \begin{cases} a = -5/13 \\ b = 1/13 \end{cases}$$

and

$$\boxed{y_p = -\frac{5}{13}\cos(x) + \frac{1}{13}\sin(x)}$$

(3) Find y_p for $y'' - 4y = 2e^{2x}$
Try $y_p = ae^{2x}$, $y_p' = 2ae^{2x}$, $y_p'' = 4ae^{2x}$

then $\underbrace{4ae^{2x} - 4ae^{2x}}_{=0} = 2e^{2x}$ ...

So ae^{2x} is not a good guess

One might wonder whether there is a systematic way to find a good guess for y_p.
If $f(x)$ is "nice" then the answer is yes.
Before we look at such cases it is interesting to state a theorem that works for all $f(x)$.

THEOREM If $F(x) = F_1(x) + F_2(x) + \dots + F_k(x)$ then $y_p = y_{p_1} + y_{p_2} + \dots + y_{p_k}$ where

$$L(y_{p_i}) = F_i(x) \quad \text{for all } i$$

The following trick can the be applied to find y_p if $F(x) = F_1(x) + F_2(x) + \dots + F_k(x)$ where

$$F_i(x) = P_m(x)\, e^{ax} \cos(bx)$$

or

$$F_i(x) = \underbrace{P_m}_{\text{Polynomial of degree } m}(x)\, e^{ax} \sin(bx)$$

i.e. $F(x)$ is a finite sum of products of a polynomial, an exponential, and a sin or cos function in x.

THEOREM If $F(x)$ is of the form described above then for every $F_i(x)$ we have that

(A) If neither $F_i(x)$, nor any of its derivatives contains a term that solves the homogeneous equation,

$$y_{p_i} = (A_0 + A_1 x + \cdots + A_m x^m) e^{ax} \cos(bx) + (B_0 + B_1 x + \cdots + B_m x^m) e^{ax} \sin(bx)$$

(B) Else

$$y_{p_i} = x^s (A_0 + A_1 x + \cdots + A_m x^m) e^{ax} \cos(bx) + x^s (B_0 + B_1 x + \cdots + B_m x^m) e^{ax} \sin(bx)$$

where s is the lowest positive integer such that y_{p_i} contains no terms that solve the associated homogeneous equation.

Note To check whether case (A) or (B) applies you need to look at the terms that appear in $F_i(x)$ AND all its derivatives $F_i'(x), F_i''(x), \ldots$

To find s, however, you only need to make sure that none of the terms of y_{p_i} solve the associated homogeneous equation. You DON'T need to check whether $y_{p_i}', y_{p_i}'', \ldots$ contain terms that solve the associated homogeneous equation.

EXAMPLES (1) Example (3) revisited: we wanted to find y_p for

$$y''-4y = 2e^{2x}$$

$f(x)$ is of the form $\underbrace{P_0(x)}_{=2}\, e^{\overset{=2}{a}x} \cos(\underbrace{b}_{=0} x)$ so we can apply our theorem.

Since e^{2x} solves $y''-4y'=0$, we're dealing with case (B) so

$$y_p = x^s \underbrace{A_0}_{} e^{2x}$$

since 3 is a polynomial of degree 0.

The solutions to $y''-4y=0$ are: $y_H = c_1 e^{2x} + c_2 e^{-2x}$ so by setting $s=1$ we see that y_p does not solve $y''-4y=0$. So

$$y_p = A_0\, x\, e^{2x}$$

$$y_p' = A_0 e^{2x} + 2A_0 x e^{2x}$$

$$y_p'' = 2A_0 e^{2x} + 2A_0 e^{2x} + 4A_0 x e^{2x}$$

$$= 4A_0 e^{2x} + 4A_0\, x\, e^{2x}$$

So $\quad 4A_0 e^{2x} + \cancel{4A_0 x e^{2x}} - \cancel{4A_0 x e^{2x}} = 2e^{2x}$

$$\Rightarrow 4A_0 = 2 \Rightarrow A_0 = \frac{1}{2} \Rightarrow \boxed{y_p = \frac{x}{2} e^{2x}}$$

(2) Solve $L(y) = y''' + y'' = \overbrace{\underbrace{3e^x}_{f_1(x)} + \underbrace{4x^2}_{f_2(x)}}^{f(x)}$

1 First we find y_H (1) CE: $r^3 + r^2 = 0 \Rightarrow r^2(r+1) = 0$

(2)

rt	val	mult
r_1	0	2
r_2	-1	1

(3) $y_H = C_1 + C_2 x + C_3 e^{-x}$

2 Now we find $y_p = y_{p_1} + y_{p_2}$

y_{p_1} 〉 e^x, nor any of its derivatives solves $L(y) = 0$

So $y_{p_1} = A_0 e^x$, $y''_{p_1} = y'''_{p_1} = A_0 e^x$

Plug it into $L(y) = 3e^x$ to get

$$2A_0 e^x = 3e^x$$

$$\Rightarrow A_0 = \frac{3}{2}$$

$$\Rightarrow \boxed{y_{p_1} = \frac{3}{2} e^x}$$

y_{p_2} 〉 Even though x^2 doesn't solve $L(y) = 0$, its derivatives do so that puts us in case (B)

$$y_{p_2} = x^s(A_0 + A_1 x + A_2 x^2)$$

where s is the smallest integer such that none of the terms of y_{p_2} solve $L(y) = 0$, i.e.

$L(A_0 x^s) \neq 0$ and $L(A_1 x^{s+1}) \neq 0$ and $L(A_2 x^{s+2}) \neq 0$

The smallest s for which this is true is $s = 2$

so

$$y_{p_2} = A_0 x^2 + A_1 x^3 + A_2 x^4$$

$$y_{p_2}' = 2A_0 x + 3A_1 x^2 + 4A_2 x^3$$
$$y_{p_2}'' = 2A_0 + 6A_1 x + 12A_2 x^2$$
$$y_{p_2}''' = 6A_1 + 24A_2 x$$

So $\quad 6A_1 + 24A_2 x + 2A_0 + 6A_1 x + 12A_2 x^2 = 4x^2$

$$\Longleftrightarrow (12A_2)x^2 + (6A_1 + 24A_2)x + 6A_1 + 2A_0 = 4x^2$$

$$\Rightarrow \begin{cases} A_2 = \frac{1}{3} \\ A_1 + 4A_2 = 0 \\ 3A_1 + A_0 = 0 \end{cases} \Rightarrow \begin{cases} A_2 = \frac{1}{3} \\ A_1 = -\frac{4}{3} \\ A_0 = 4 \end{cases}$$

$$\Rightarrow \qquad \boxed{y_{p_2} = 4x^2 - \frac{4}{3}x^3 + \frac{1}{3}x^4}$$

All together we find that

$$\boxed{y = y_H + y_P = y_H + y_{P_1} + y_{P_2} = C_1 + C_2 x + C_3 e^{-x} + \frac{3}{2}e^{x} + 4x^2 - \frac{4}{3}x^3 + \frac{1}{3}x^4}$$

Remember These rules only work if $F(x)$ is of a nice form.

But what if, e.g., $F(x) = \tan(x)$, or $F(x) = \ln(x)$?

There is another method, called VARIATION OF PARAMETERS that, in theory, works for every $F(x)$. It even works when the coefficients of the LnDE are not constant.

We will explain it only for the case of an L2DE, of the following form $y'' + P(x)y' + Q(x)y = f(x)$

Let $y_H = C_1 y_1 + C_2 y_2$. The idea is to set

$$y_P = u_1 y_1 + u_2 y_2$$ where u_1, u_2 are functions of x.

If we would plug this into the DE then we get 1 linear DE in 2 variables: u_1, u_2. This will always have an ∞ # of solutions. Since we only want 1 solution, we'll add an artificial constraint on u_1 and u_2 that is designed to make the task of finding u_1, u_2 easier and still guarantees a single solution.

Since
$$y_P = u_1 y_1 + u_2 y_2$$
$$y_P' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$$

We now set $\boxed{u_1' y_1 + u_2' y_2 = 0}$ (E1)

so
$$y_P' = u_1 y_1' + u_2 y_2'$$

and
$$y_P'' = u_1' y_1' + u_1 y_1'' + u_2' y_2 + u_2 y_2''$$

so

$$u_1' y_1' + u_1 y_1 + u_2' y_2' + u_2 y_2'' + P(x)(u_1 y_1' + u_2 y_2') + Q(x)(u_1 y_1 + u_2 y_2) = F(x)$$

$$\Leftrightarrow u_1' y_1' + u_2' y_2' + u_1(y_1'' + \cancel{P(x) y_1' + Q(x)} y_1) + u_2(y_2'' + \cancel{P(x) y_2' + Q(x)} y_2) = F(x)$$

$$\Leftrightarrow \quad \boxed{u_1' y_1' + u_2' y_2' = F(x)} \quad (E_2)$$

E1 and E2 form a linear system of equations in the variables u_1', u_2':

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 & (E_1) \\ u_1' y_1' + u_2' y_2 = F(x) & (E_2) \end{cases}$$

This system has a solution only if $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W(y_1, y_2) \neq 0$

Since y_1, y_2 are LI this is always true!

$$(E_2)\cdot y_1 - (E_1)\cdot y_1' : \; u_2' y_2' y_1 - u_2' y_2 y_1' = F(x) y_1$$

$$\Rightarrow u_2' = \frac{f(x) y_1}{W(y_1,y_2)} \Rightarrow \boxed{u_2 = \int \frac{F(x) y_1}{W(y_1,y_2)} dx}$$

$$(E_2)\cdot y_2 - (E_1)\cdot y_2' : \; u_1'(y_1' y_2 - y_1 y_2') = F(x) y_2$$

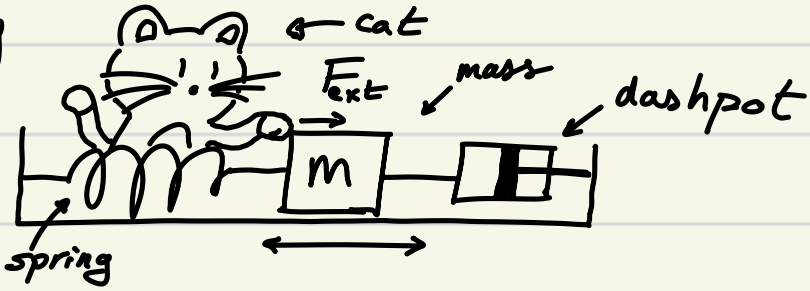
$$\Rightarrow u_1' = -\frac{f(x) y_2}{W(y_1,y_2)} \Rightarrow \boxed{u_1 = -\int \frac{F(x) y_2}{W(y_1,y_2)} dx}$$

So

$$\boxed{y_P = -y_1 \int \frac{F(x) y_2}{W(y_1,y_2)} dx + y_2 \int \frac{F(x) y_1}{W(y_1,y_2)} dx}$$

SEC 3.6. FORCED OSCILLATIONS & RESONANCE

Consider a spring-dashpot system where an external force is acting on the object attached to the spring



Newton's second law tells us that for a constant mass

$$F_{total} = m\frac{d^2x}{dt^2}$$

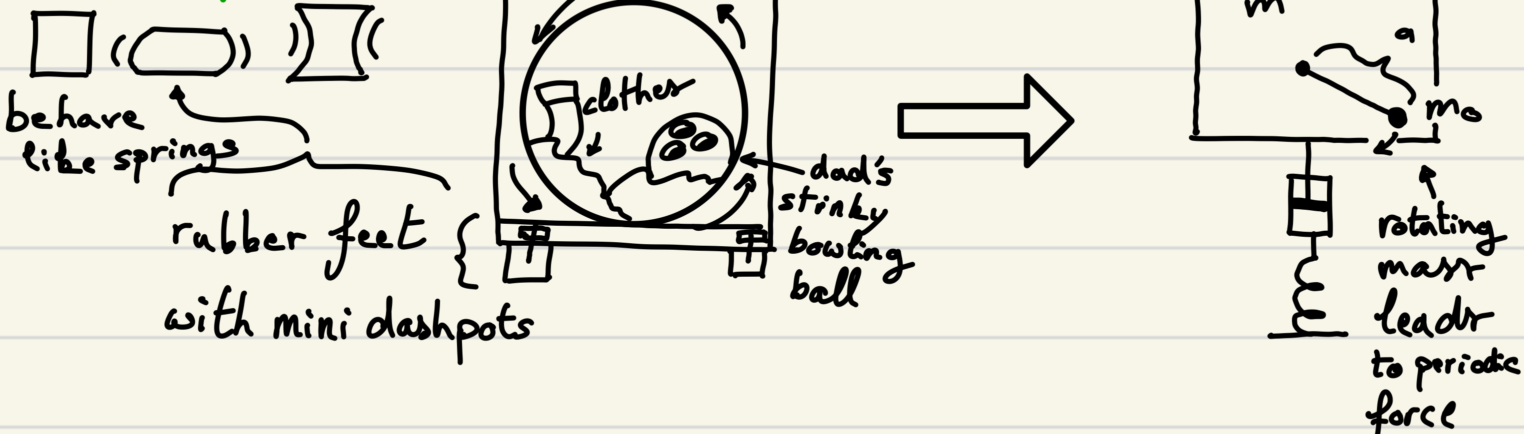
so

$$F_{ext} + F_{spring} + F_{dashpot} = m\frac{d^2x}{dt^2}$$

$$\Rightarrow \quad F_{ext} - kx - c\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

$$\Rightarrow \quad \boxed{m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_{ext}}$$

EXAMPLE



In this section we will focus on the case where $F_{ext} = F_0 \cos(\omega t)$ (with $F_0 \neq 0$)

Undamped forced oscillations

$$L(x) = mx'' + kx = F_0 \cos(\omega t)$$

We now know how to solve this equation:

$$x(t) = x_H(t) + x_p(t)$$

For the undamped case:

$$X_H = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

with $\omega_0 = \sqrt{\frac{k}{m}}$

There are two scenarios for x_p:

(a) $\omega \neq \omega_0$ so $F_0 \cos(\omega_0 t)$, nor any of its derivates solve $L(x) = 0$

(b) $\omega = \omega_0$ and $F_0 \cos(\omega_0 t)$ solves $L(x) = 0$

(a) $\underline{\omega \neq \omega_0}$ Then $x_p = a\cos(\omega t) + b\sin(\omega t)$

$$= aC_\omega + bS_\omega$$

where $C_\omega' = -\omega S$, $S_\omega' = \omega C_\omega$.

Then

$$x_p = aC_\omega + bS_\omega$$

$$x_p'' = -\omega^2 a C_\omega - \omega^2 b S_\omega$$

so

$$C_\omega(ka - m\omega^2 a) + S_\omega(kb - m\omega^2 b) = F_0 C_\omega$$

$$\Rightarrow b = 0 \quad \text{and} \quad a = \frac{F_0}{k - m\omega^2} = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

$$\Rightarrow \qquad x = A\cos(\omega_0 t) + B\sin(\omega_0 t) + \frac{F_0}{m(\omega^2 - \omega_0^2)}\cos(\omega t)$$

or

$$X = C\cos(\omega_0 t - \alpha) + \frac{F_0}{m(\omega^2 - \omega_0^2)}\cos(\omega t)$$

In the special senario where $x(0) = 0$ and $x'(0) = 0$ we have that

$$\begin{cases} 0 = C\cos(\alpha) + \dfrac{F_0}{m(\omega^2 - \omega_0^2)} \quad \Rightarrow C \neq 0 \\ 0 = -C\omega_0 \sin(\alpha) \end{cases}$$

so $\quad \alpha = 0, \quad C = -\dfrac{F_0}{m(\omega^2 - \omega_0^2)}$

$$\Rightarrow \qquad x = \frac{F_0}{m(\omega^2 - \omega_0^2)}\left(\cos(\omega t) - \cos(\omega_0 t)\right)$$

By using 'addition of angle' formulas we can simplify this result further

(1) $\quad \cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

(2) $\quad \cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$

so (2)-(1) gives

$$\cos(\underbrace{a-b}_{\omega t}) - \cos(\underbrace{a+b}_{\omega_0 t}) = 2\sin(a)\sin(b)$$

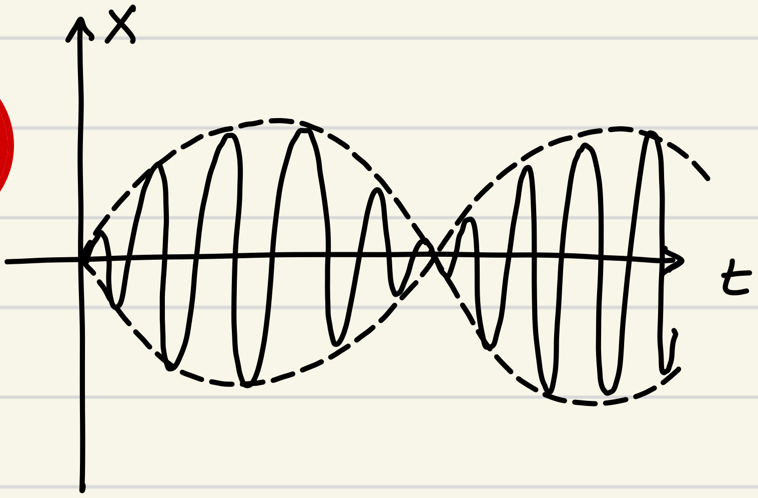
$$\begin{cases} \omega t = a - b \\ \omega_0 t = a + b \end{cases} \Rightarrow \begin{cases} a = \dfrac{\omega_0 t + \omega t}{2} \\ b = \dfrac{\omega_0 t - \omega t}{2} \end{cases}$$

so $\cos(\omega t) - \cos(\omega_0 t) = 2\sin\left(\frac{\omega_0+\omega}{2}t\right)\sin\left(\frac{\omega_0-\omega}{2}t\right)$

and

$$x = \frac{2F_0}{m(\omega^2-\omega_0^2)}\sin\left(\frac{\omega_0+\omega}{2}t\right)\sin\left(\frac{\omega_0-\omega}{2}t\right)$$

If ω is close to ω_0 then $|\omega_0+\omega| >> |\omega_0-\omega|$ and we have a product of two factors, 1 fast oscillating 1 slow oscillating.



This phenomenon was researched extensively in Dre's doctorate where he called it beats. [Source Missing]

Case (b) $\omega = \omega_0$

In this case $F_0\cos(\underset{\underset{\omega_0}{\shortparallel}}{\omega} t)$ solves $L(x)=0$ so we need to use

$$x_p = t^s(A\cos(\omega_0 t)+B\sin(\omega_0 t))$$

For $s=1$ x_p contains no term that solves $L(x)=0$.

We set $C_\omega = \cos(\omega_0 t)$ $S_\omega = \sin(\omega_0 t)$

so $C_\omega' = -\omega_0 S_\omega$, $S_\omega' = \omega_0 C_\omega$

So $x_p = AtC_\omega + BtS_\omega$

$$x_p' = AC_\omega - A\omega_0 t S_\omega + BS_\omega + B\omega_0 t C_\omega$$

$$x_p'' = -A\omega_0 S_\omega - A\omega_0 S_\omega - A\omega_0^2 t C_\omega + B\omega_0 C_\omega + B\omega_0 C_\omega - B\omega_0^2 t S_\omega$$

$$= -2A\omega_0 S_\omega + 2B\omega_0 C_\omega - A\omega_0^2 t C_\omega - B\omega_0^2 t S_\omega$$

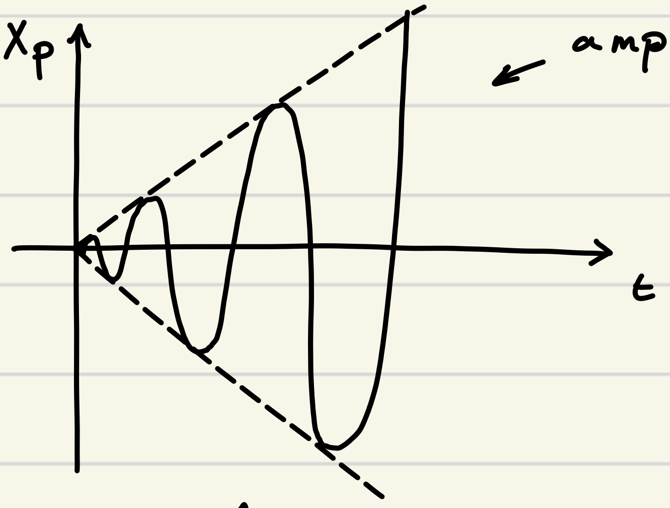
$$\Rightarrow -2mA\omega_0 S_\omega + 2mB\omega_0 C_\omega - mA\omega_0^2 t C_\omega - mB\omega_0^2 t S_\omega + kAtC_\omega + kBtS_\omega = F_0 C_\omega$$

$$\Rightarrow S_\omega(-2mA\omega_0) + C_\omega(2mB\omega_0) + tS_\omega(kB\cancel{-m\omega_0^2 B}) + tC_\omega(\cancel{kA-m\omega_0^2}A) = F_0 C_\omega$$

since $\omega_0^2 = \frac{k}{m}$

$$\Rightarrow \begin{cases} -2mA\omega_0 = 0 \\ 2mB\omega_0 = F_0 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = \dfrac{F_0}{2m\omega_0} \end{cases}$$

and $$x = C\cos(\omega_0 t - \alpha) + \underbrace{\frac{F_0}{2m\omega_0} t\cos(\omega_0 t)}_{x_p}$$



amplitude $\xrightarrow[t\to\infty]{} \infty$

resonance

FORCED DAMPED OSCILLATIONS

The DE is now of the following form:

$$L(x) = m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_0 \cos(\omega_0 t)$$

There were 3 scenarios for the solutions of $L(x)=0$

1) Overdamped system

$$x_H = c_1 e^{at} + c_2 e^{bt} \quad \text{with } a, b < 0$$

2) Critically damped system

$$x_H = (c_1 + c_2 t) e^{at} \quad \text{with } a < 0$$

3) Underdamped system

$$x_H = e^{at}(c_1 \cos(bt) + c_2 \sin(bt)) \quad a < 0$$

In all 3 scenarios we see that x_H decays exponentially fast. This has several consequences:

1) x_p nor any of its derivatives shares a term with x_H

2) $X = x_H + x_p \approx \underbrace{x_p}_{\to \text{Transient solution}}$ for large t

Lets find x_p

$$X_p = a\cos(\omega t) + b\sin(\omega t)$$

$$= aC_\omega + bS_\omega \qquad \text{with } C_\omega' = -\omega S_\omega,\ S_\omega' = \omega C_\omega$$

$$X_p' = -a\omega S_\omega + b\omega C_\omega$$

$$X_p'' = -a\omega^2 C_\omega - b\omega^2 S_\omega$$

so

$$-ma\omega^2 C_\omega - mb\omega^2 S_\omega - ca\omega S_\omega + cb\omega C_\omega + kaC_\omega + kbS_\omega = F_0 C_\omega$$

$$\Rightarrow \quad C_\omega\left(a\underbrace{(k-m\omega^2)}_{D} + bc\omega\right) + S_\omega\left(b(k-m\omega^2) - ac\omega\right) = F_0 C_\omega$$

$$\Rightarrow \begin{cases} a\, m(\omega_0^2-\omega^2) + b(c\omega) = F_0 \\ -ac\omega + b\underbrace{m(\omega_0^2-\omega^2)}_{D} = 0 \end{cases} \Rightarrow \begin{cases} aD + bc\omega = F_0 \\ -ac\omega + bD = 0 \end{cases}$$

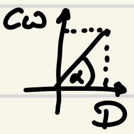
$$\Rightarrow \begin{cases} b\dfrac{D^2}{c\omega} + bc\omega = F_0 \\ a = b\dfrac{D}{c\omega} \end{cases} \Rightarrow \begin{cases} b = \dfrac{F_0 c\omega}{D^2+c^2\omega^2} \\ a = \dfrac{F_0 D}{D^2+c^2\omega^2} \end{cases}$$

so

$$X_p = \frac{F_0}{D^2+c^2\omega^2}\left(D\cos(\omega t) + c\omega\sin(\omega t)\right)$$

$$= \frac{F_0}{D^2+c^2\omega^2}\sqrt{D^2+c^2\omega^2}\cos(\omega t - \alpha)$$

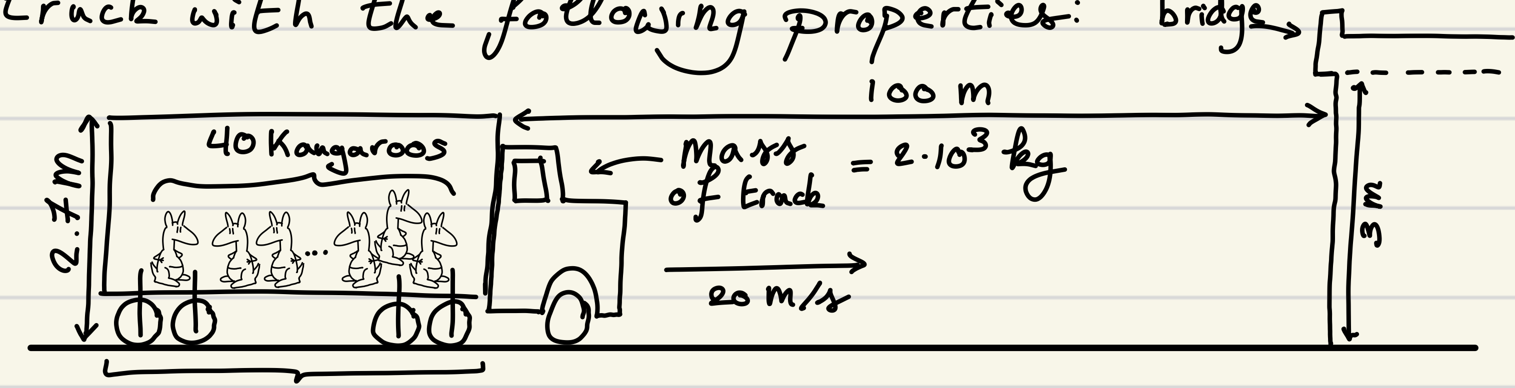
$\uparrow$ angle



$$= \frac{F_0}{\sqrt{D^2+c^2\omega^2}}\cos(\omega t - \alpha)$$

While there is no actual resonance for any value of ω the amplitude of the motion does depend on ω. If there is a value $\omega_r > 0$ for ω that maximizes the amplitude then we say that there is practical resonance for $\omega = \omega_r$.

EXAMPLE A package delivery service uses a truck with the following properties:



8 wheels, each with spring-daspot system with
$k = 8 \cdot 10^4 \frac{N}{m}$, $c = 3 \cdot 10^3 \frac{N \cdot s}{m}$

The truck is delivering 40 kangaroos of 50 kg each to a certain Lord Mr. Dr. Prof. G. V. At a certain moment the kangaroos notice a bridge of height 3 m at a distance of 100 m from the edge of the body of the truck, and start exerting a force

$$F_0 \cos(\sqrt{160}\, t)$$

with the aim to raise the body of the truck high enough to hit the bridge.
If quora is correct and a kangaroo can exert a force of 3400 N with its kicks, will they succeed?

SOLUTION

8 wheels carry 2000 kg + 40 kangaroos $\cdot 50 \frac{\text{kg}}{\text{kangaroo}} = 4000$ kg
Therefore, each wheel carries 500 kg.
The DE is thus

$$500\, x'' + 3\cdot 10^3\, x' + 8\cdot 10^4 = 3300 \cos(\sqrt{160}\, t)$$

Step 1 Find x_H

CE: $500 r^2 + 3\cdot 10^3 r + 8\cdot 10^4 = 0$

$$\Rightarrow \quad r = \frac{-3\cdot 10^3 \pm \sqrt{9\cdot 10^6 - 4\cdot 500\cdot 8\cdot 10^4}}{10^3} = -3 \pm i\sqrt{151}$$

$$x_H = e^{-3t}\left(a\cos(\sqrt{151}\, t) + b\sin(\sqrt{151}\, t)\right)$$

$$x_p = \frac{F_0}{\sqrt{D^2 + c^2\omega^2}} \cos(\omega t - \alpha)$$

$$= 4.86\cdot 10^{-3} \cos(\sqrt{160}\, t - 2.35) \leftarrow \text{pretty much nothing}$$

Using Mathematica to solve $x(0) = 0$, $x'(0) = 0$ for a, b gives us

$$x = e^{-3t}\left(0.34\cos(\sqrt{151}\, t) + 0.072\sin(\sqrt{151}\, t)\right) + 4.86\cdot 10^{-3}\cos(\sqrt{160}\, t - 2.35)$$

So at $t = \frac{100\,\text{m}}{25\,\frac{\text{m}}{\text{s}}} = 4\,\text{s}$ the max deviation from the equilibrium position is $< \frac{0.34}{e^{12}} + \frac{0.072}{e^{12}} + \frac{4.86}{1000} \ll 0.3$

and the kangaroos won't escape !!

CHAPER 4: SYSTEMS OF ODEs

SEC 4.1. FIRST ORDER SYSTEMS & APPLICATIONS.

So far we always solved 1 equation in 1 unknown function. There are, however, many situations where you need to solve n equations in n unknown functions.

EXAMPLES Newton's 2nd law for an object with constant mass in 3D is

$$\begin{cases} m\dfrac{d^2x}{dt^2} = F_x \\ m\dfrac{d^2y}{dt^2} = F_y \\ m\dfrac{d^2z}{dt^2} = F_z \end{cases}$$

where F_x, F_y, F_z are functions that can depend on time, positi velocity, ...

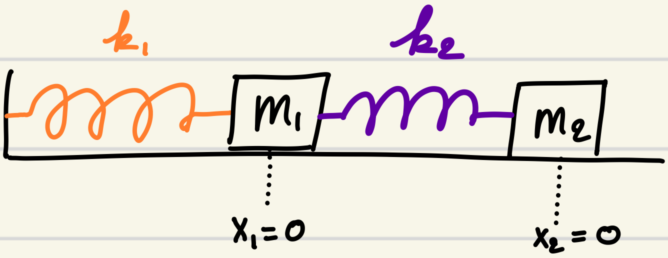
The solution to such system is a triple of functions $(x(t), y(t), z(t))$ such that if you plug these into the system you get

$$\begin{cases} 0 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

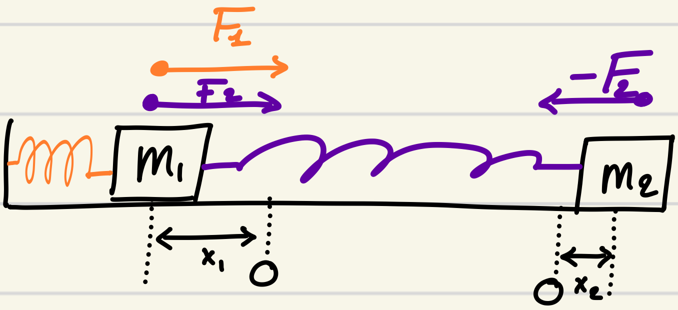
For this course we will restrict ourselves to 1^{st} order systems with as many equations as variables.

EXAMPLES (1) Two springs attached to each other.

In rest such system looks like



When we stretch/compress the strings, they will apply forces to the objects



For m_1: $F_{total_1} = F_1 + F_2 = -x_1 k_1 + (x_2 - x_1) k_2 = m_1 x_1''$

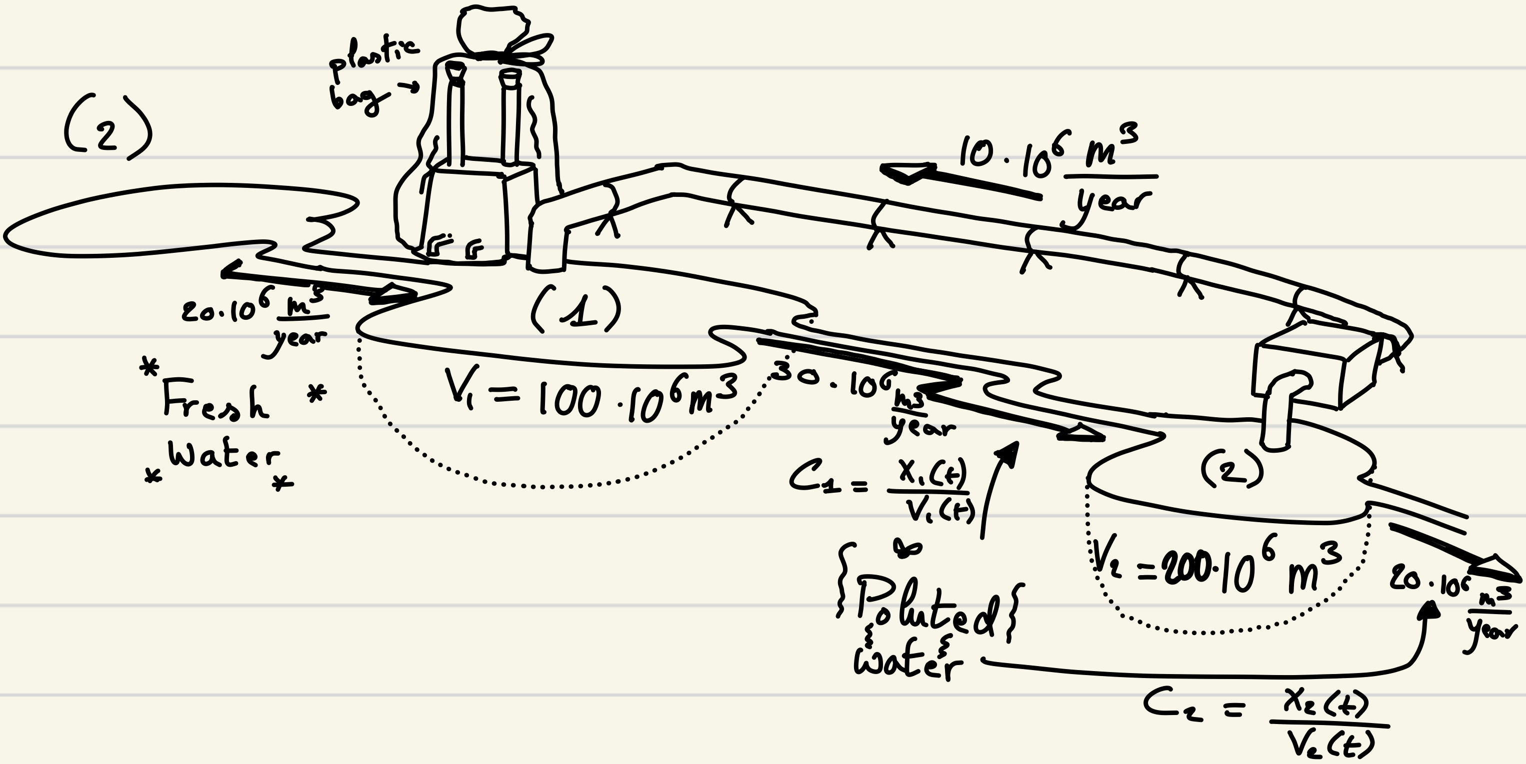
$F_{total_2} = -F_2 = (x_1 - x_2) k_2 = m_2 x_2''$

So
$$\begin{cases} x_1'' = \dfrac{(-k_1 - k_2)}{m_1} x_1 + \dfrac{k_2}{m_1} x_2 \\ x_2'' = \dfrac{k_2}{m_1} x_1 - \dfrac{k_2}{m_2} x_2 \end{cases}$$

We can transform this system to a first order one by setting $x_3 = x_1'$, $x_4 = x_2'$ (and thus $x_3' = x_1''$, $x_4' = x_2''$)

$$\begin{cases} x_3' = -\dfrac{k_1 + k_2}{m_1} x_1 + \dfrac{k_3}{m_1} x_2 \\ x_1' = x_3 \\ x_4' = \dfrac{k_2}{m_1} x - \dfrac{k_2}{m_2} x_2 \\ x_2' = x_4 \end{cases}$$

So we have 4 equations in 4 unknown functions x_1, x_2, x_3, x_4 each in 1 variable: t.



Let x_1 be mass of polutant in lake 1

x_2 " " " " 2

Then

$$\begin{cases} X_1'(t) = C_2(t) \cdot 10 \cdot 10^6 - C_1(t)\, 30 \cdot 10^6 \\ X_2'(t) = C_1(t) \cdot 30 \cdot 10^6 - C_2(t)\, 10 \cdot 10^6 - C_2(t) \cdot 20 \cdot 10^6 \end{cases}$$

Since $V_1 = 100 \cdot 10^6$ and $V_2 = 200 \cdot 10^6$ are constants, we have that $C_1 = \frac{x_1}{100 \cdot 10^6}$, $C_2 = \frac{x_2}{200 \cdot 10^6}$

so

$$\begin{cases} X_1'(t) = -\frac{3}{10} x_1 + \frac{1}{20} x_2 \\ X_2'(t) = \frac{3}{10} x_1 - \frac{3}{20} x_2 \end{cases}$$

Often (especially in physics) one encounters systems of higher order DEs. These can always be reduced to an equivalent system of 1^{st} order DEs. We will show how to do this for a single equation. To convert a whole system, you just need to apply this method to each equation.

EXAMPLE (1) Convert $x^{(n)} = f(x^{(n-1)}, x^{(n-2)}, \dots, x', x, t)$ to a system of first order DEs.

Solution: Define $x_i = x^{(i)}$

then we get the following system of DE's:

$$\boxed{\begin{cases} x_0' = x_1 \\ x_1' = x_2 \\ \vdots \\ x_{n-2}' = x_{n-1} \\ x_{n-1}' = f(x_{n-1}, x_{n-2}, \dots, x_1, x_0, t) \end{cases}}$$

(2) Convert $x'''' - 6x'' - x = t^2$ to a system of 1^{st} order DEs

Solution

Set $x_1 = x$, $x_2 = x'$, $x_3 = x''$, $x_4 = x'''$.

Then we get $\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_4 \\ x_4' - 6x_3 - x_1 = t^2 \end{cases} \Rightarrow \boxed{\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_4 \\ x_4' = 6x_3 + x_1 + t^2 \end{cases}}$

(3) Convert the following system of DEs to a first order one

$$\begin{cases} x'' = 3x + y \\ y'' = 2x' + x - y + t^2 \end{cases}$$

Solution

Lets do this equation by equation.

For $x'' = 3x + y$ we set $x_1 = x$, $x_2 = x'$, and $y_1 = y$ so we get

$$\begin{cases} x_1' = x_2 \\ x_2' = 3x_1 + y_1 \end{cases}$$

For the second equation, $y'' = 2x' + x - y + t^2$ we already set $x' = x_2$, $x = x_1$, $y = y_1$. We only need to set $y_2 = y'$ to get

$$\begin{cases} y_1' = y_2 \\ y_2' = 2x_2 + x_1 - y_1 + t^2 \end{cases}$$

All together we have the following system

$$\boxed{\begin{cases} x_1' = x_2 \\ x_2' = 3x_1 + y_1 \\ y_1' = y_2 \\ y_2' = 2x_2 + x_1 - y_1 + t^2 \end{cases}}$$

In some cases one can also extract DEs in a single variable from a system of ODEs.

EXAMPLE Derive an ODE in the variable x from the following system

$$\begin{cases} x' = -2y & ① \\ y' = \frac{x}{2} & ② \end{cases}$$

Solution

By deriving both sides of ① we get $x'' = -2y'$

Since $y' = \frac{x}{2}$, this means that

$$x'' = -2\frac{x}{2} = -x$$

So $\boxed{x'' + x = 0}$

It is important to note that, by taking a derivative of both sides of an equation and then solving that equation we might have introduced new solutions. Take, e.g., the DE $x' = 2$. This equation has $x = 2t + c$ as a solution. If we derive both sides of this equation then we get $x'' = 0$. This has $x = c_1 t + c_2$ as a solution! So we get an ∞ number of extra solutions! This is because the derivative is not an invertible operator: applying it to a function causes you to lose information! After deriving an equation and solving it, you should always substitute your solution into the original equation to find any possible constraints you might have missed!

For $x' = 2$, substituting $x = c_1 t + c_2$ reveals that $c_1 = 2$ so the actual solution is $x = 2t + c_2$.

SEC 4.2 THE METHOD OF ELIMINATION

PREVIOUS SECTION:

higher order DE: $x^{(n)} = f(x^{(n-1)}, \dots, x'', x', x, t)$

$$\Downarrow$$

System of 1st order DE's
$$\begin{cases} x_0' = x_1 \\ x_1' = x_2 \\ \vdots \\ x_{n-2}' = x_{n-1}' \\ x_{n-1}' = f(x_{n-1}, \dots, x_2, x_1, x_0, t) \end{cases}$$

THIS SECTION

system of 1st order DE's

$$\Downarrow$$

higher order DE's in 1 var

The general technique to do this is via elimination of variables

EXAMPLE

Solve

$$\begin{cases} x' = -2y & \text{eqn } 1 \\ y' = \frac{1}{2}x & \text{eqn } 2 \end{cases}$$

If we derive eqn 1 w.r.t. x we get that

$$x'' = -2y' = -2\cdot\left(\frac{1}{2}x\right)$$

so $x'' = -x \iff x'' + x = 0 \Rightarrow x = c_1\cos(t) + c_2\sin(t)$

$$= A\cos(t-\alpha)$$

From eqn 1 we find that

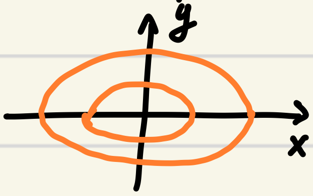
$$y = -\frac{1}{2}x' = -\frac{1}{2}\left(A\left(-\sin(t-\alpha)\right)\right)$$

$$= \frac{A}{2}\sin(t-\alpha)$$

Note since $\sin^2(\theta) + \cos^2(\theta) 1$ we have that

$$\left(\frac{2}{A}y\right)^2 + \left(\frac{1}{A}x\right)^2 = 1$$

which is the equation of an elipse with radii $r_1 = A, r_2 = \frac{A}{2}$



IMPORTANT NOTE

After finding that $x = A\cos(t-\alpha)$ we could also have derived y by using eqn 2 as follows:

$$y' = \frac{1}{2}x = \frac{A}{2}\cos(t-\alpha)$$

Integration then gives

$$y = \frac{A}{2}\sin(t-\alpha) + c$$

We could've also derived eqn 2 to obtain

$$y'' = \frac{1}{2}x' = -\frac{2}{2}y = -y$$

and thus $$y = \underline{A_2} \cos(t - \underline{\alpha_2}).$$

So depending on which method we use, we get 2, 3, or 4 integration constants for our solutions!
It turns out that these extra constants are imposters: they are related to each other.
Indeed, if we substitute

$$x = A\cos(t-\alpha)$$
$$y = \frac{A}{2}\sin(t-\alpha) + C$$

into our system we find from eqn 1 that

$$-A\sin(t-\alpha) = -2\left(\frac{A}{2}\sin(t-\alpha) + c\right)$$

$$\Leftrightarrow \quad 0 = -2c$$

so $$\boxed{c=0}$$

If we substitute $x = A\cos(t-\alpha)$

$$y = A_2 \sin(t-\alpha_2)$$

into the system we find

$$-A\cos(t-\alpha) = -2A_2\cos(t-\alpha_2)$$
$$A_2\cos(t-\alpha_2) = \frac{A}{2}\cos(t-\alpha)$$
$$\Leftrightarrow \quad A_2\cos(t-\alpha) = 2A\cos(t-\alpha_2) \qquad \text{for all } t$$

After expanding $A\cos(t-\alpha) = c_1\cos(t) + c_2\sin(t)$

$$A_2\cos(t-\alpha_2) = c_3\cos(t) + c_4\sin(t)$$

we get

$$2c_1\cos(t) + 2c_2\sin(t) = c_3\cos(t) + c_4\sin(t)$$

Setting $t=0$ gives $2c_1 = c_3$

Setting $t=\frac{\pi}{2}$ gives $2c_2 = c_4$

so there are actually only 2 independent integration constants.

CHAPTER 5 LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

SEC 5.1 MATRICES & LINEAR SYSTEMS

REVIEW OF NOTATION & TERMINOLOGY

An $\underbrace{m \times n}_{\text{dimensions of } \underline{\underline{A}}}$ matrix $\underline{\underline{A}}$ is a rectangular array of numbers

$$\underline{\underline{A}} = \underset{\longleftarrow n \text{ columns} \longrightarrow}{\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ & \vdots & & \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}} \Big\updownarrow m \text{ rows}$$

Other notations are

$$\underline{\underline{A}} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ & \vdots & & \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix}$$

or

$$\underline{\underline{A}} = \boxed{\begin{matrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ & \vdots & & \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{matrix}}$$

← I'm the only one in the world who uses this notation...

If $n = 1$, so

$$\underline{\underline{A}} = \boxed{\begin{matrix} A_{11} \\ A_{21} \\ \vdots \\ A_{m1} \end{matrix}}$$

we also write this as

$$\underline{A} = \boxed{\begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_3 \end{matrix}}$$

and call $\underline{A}$ a column vector.

If $m = 1$, i.e.

$$\underline{\underline{A}} = \boxed{A_{11}\ A_{12}\ \cdots\ A_{1n}}$$

we also write

$$\underline{A} = \boxed{A_1\ A_2\ \cdots\ A_n}$$

and call $\underline{A}$ a row vector.

The elements of a matrix are denoted by $\overset{\substack{\text{only when}\\ \text{used inside the matrix}}}{\overset{\downarrow}{A_{ij}}}$ or $[\underline{\underline{A}}]_{ij}$.

Two matrices, $\underline{\underline{A}}$ and $\underline{\underline{B}}$ with the same dimensions can be added to each other to give a matrix $\underline{\underline{C}}$ with elements

$$[\underline{\underline{C}}]_{ij} = [\underline{\underline{A}} + \underline{\underline{B}}]_{ij} = [\underline{\underline{A}}]_{ij} + [\underline{\underline{B}}]_{ij}$$

Any matrix A can be multiplied with a number, say α. The newly obtained matrix $\underline{\underline{B}} = \alpha \underline{\underline{A}}$ has elements

$$[\underline{\underline{B}}]_{ij} = \alpha\, [A]_{ij}$$

An $m \times n$ matrix $\underline{\underline{A}}$ can be multiplied on the right by an $k \times l$ matrix $\underline{\underline{B}}$ ONLY if $n = k$. The result, say $\underline{\underline{C}}$, is an $m \times l$ matrix with elements

$$[\underline{\underline{C}}]_{ij} = [\underline{\underline{A}}\,\underline{\underline{B}}]_{ij} = \sum_{p=1}^{n} [\underline{\underline{A}}]_{ip} [\underline{\underline{B}}]_{pj}$$

so the columns of $\underline{\underline{C}}$ can be interpreted as linear combinations of the columns of $\underline{\underline{A}}$. Indeed, fix a column index j, then

$$[\underline{\underline{C}}]_{ij} = \sum_{p=1}^{n} \underbrace{b_p}_{=[\underline{\underline{B}}]_{pj}} [\underline{\underline{A}}]_{ip}$$

Likewise the rows of $\underline{\underline{C}}$ can be interpreted as linear combinations of rows of $\underline{\underline{B}}$. Indeed, if we fix a row index i, then

$$[\underline{\underline{C}}]_{ij} = \sum_{p=1}^{n} \underbrace{a_p}_{=[\underline{\underline{A}}]_{ip}} [\underline{\underline{B}}]_{pj}$$

visual aid

EXAMPLES

(1) Let

$$\underset{3\times 2}{\underline{\underline{A}}} = \begin{array}{|c:c|} \hline 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ \hline \multicolumn{1}{c}{\uparrow} & \multicolumn{1}{c}{\uparrow} \\ \multicolumn{1}{c}{\underline{c_1}} & \multicolumn{1}{c}{\underline{c_2}} \end{array}, \quad \underset{2\times 3}{\underline{\underline{B}}} = \begin{array}{|ccc|} \hline 0 & 1 & 2 \\ 3 & 4 & 5 \\ \hline \end{array}$$

Then

$$\underline{\underline{A}}\;\underline{\underline{B}} = \begin{array}{|c:c:c|} \hline 0\,\underline{c_1} + 3\,\underline{c_2} & 1\,\underline{c_1} + 4\,\underline{c_2} & 2\,\underline{c_1} + 5\,\underline{c_2} \\ \hline \end{array}$$

$$= \begin{array}{|c:c:c|} \hline 6 & 9 & 12 \\ 12 & 19 & 26 \\ 18 & 29 & 40 \\ \hline \end{array}$$

(2) Let

$$\underset{3\times 3}{\underline{\underline{A}}} = \begin{array}{|c:c:c|} \hline A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \\ \hline \multicolumn{1}{c}{\uparrow} & \multicolumn{1}{c}{\uparrow} & \multicolumn{1}{c}{\uparrow} \\ \multicolumn{1}{c}{\underline{c_1}} & \multicolumn{1}{c}{\underline{c_2}} & \multicolumn{1}{c}{\underline{c_3}} \end{array}, \quad \underset{3\times 1}{\underline{x}} = \begin{array}{|c|} \hline x_1 \\ x_2 \\ x_3 \\ \hline \end{array}$$

then $$\underline{\underline{A}} \cdot \underline{x} = x_1 \underline{c_1} + x_2 \underline{c_2} + x_3 \underline{c_3} = \underbrace{\boxed{\begin{matrix} x_1 A_{11} & + x_2 A_{12} & + x_3 A_{13} \\ x_1 A_{21} & + x_2 A_{22} & + x_3 A_{23} \\ x_1 A_{31} & + x_2 A_{32} & + x_3 A_{33} \end{matrix}}}_{\text{column vector}}$$

You can verify the following properties of matrix multiplication:

1. $\underline{\underline{A}}\,(\underline{\underline{B}}\,\underline{\underline{C}}) = (\underline{\underline{A}}\,\underline{\underline{B}})\,\underline{\underline{C}}$ (associative)
2. There are matrices $\underline{\underline{A}}, \underline{\underline{B}}$ s.t. $\underline{\underline{A}}\,\underline{\underline{B}} \neq \underline{\underline{B}}\,\underline{\underline{A}}$ (not commutativ
3. $\underline{\underline{A}}\,(\underline{\underline{B}} + \underline{\underline{C}}) = \underline{\underline{A}}\,\underline{\underline{B}} + \underline{\underline{A}}\,\underline{\underline{C}}$ (distributive)
   $(\underline{\underline{A}} + \underline{\underline{B}})\,\underline{\underline{C}} = \underline{\underline{A}}\,\underline{\underline{C}} + \underline{\underline{B}}\,\underline{\underline{C}}$
4. $\underline{\underline{A}}\,\underline{\underline{B}}$ can be $\boxed{\begin{matrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \\ \vdots & & & \\ 0 & 0 & & 0 \end{matrix}} = \underline{\underline{0}}$ without $\underline{\underline{A}} = \underline{\underline{0}}$ or $\underline{\underline{B}} = \underline{\underline{0}}$
5. The matrix $\underset{n\times n}{\underline{\underline{1}}} = \boxed{\begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 1 \end{matrix}}$ satisfies $\underset{m\times n}{\underline{\underline{A}}}\,\underline{\underline{1}} = \underline{\underline{A}}$ and $\underline{\underline{1}}\,\underset{n\times m}{\underline{\underline{B}}} = \underline{\underline{B}}$
6. An $m\times m$ matrix $\underline{\underline{A}}$ has an inverse $\underline{\underline{A}}^{-1}$ if and only if $\det(\underline{\underline{A}}) \neq 0$. If so, it satisfies $\underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{A}}^{-1}\underline{\underline{A}} = \underset{m\times m}{\underline{\underline{1}}}$

MATRIX VALUED FUNCTIONS

A matrix whose elements are functions is called a matrix valued function.

EXAMPLES (1)

$$\underline{x}(t) = \boxed{\begin{matrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{matrix}}$$

(2)

$$\underline{\underline{A}}(t) = \boxed{\begin{matrix} A_{11}(t) & A_{12}(t) & \cdots & A_{1n}(t) \\ & \vdots & & \\ A_{m1}(t) & A_{m2}(t) & \cdots & A_{mn}(t) \end{matrix}}$$

Such functions can be differentiated as follows:

$$\left[\underline{\underline{A}}'(t)\right]_{ij} = \left[\frac{d}{dt}\underline{\underline{A}}(t)\right]_{ij} = \frac{d}{dt}\left(\left[\underline{\underline{A}}\right]_{ij}\right)$$

FIRST ORDER LINEAR SYSTEMS

The nice systems for which the theorem of existence and uniqueness applies can be written in the form

$$\underline{x}'(t) = \underline{\underline{M}}(t)\,\underline{x}(t) + \underline{f}(t) \qquad \circledast$$

Where $\underline{x}, \underline{x}', \underline{f}$ are vectors of length n, and $\underline{\underline{M}}$ is an $n \times n$ matrix.

A solution to $\circledast$ is a vector $\underline{x}(t) = \boxed{\begin{matrix} x_1(t) \\ \vdots \\ x_n(t) \end{matrix}}$ that satisfies $\circledast$.

$\circledast$ is called homogeneous if $\underline{f}(t) = \underline{0} = \boxed{\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}}$

THEOREM If $\circledast$ is homogeneous and the vectors $\underline{x}_1, \dots, \underline{x}_n$ are linearly independent solutions then all solutions to $\circledast$ are of the form

$$\underline{x}(t) = c_1\,\underline{x}_1(t) + c_2\,\underline{x}_2(t) + \cdots + c_n\,\underline{x}_n(t).$$

You can check linear independence via the determinant:
$\underline{x}_1, \dots, \underline{x}_n$ are LI for all t in an interval I if and only if $\det\left(\boxed{\begin{array}{c:c:c:c} \underline{x}_1 & \underline{x}_2 & \cdots & \underline{x}_n \end{array}}\right) \neq 0$ on I

THEOREM All solutions to the non-homogeneous system $\circledast$ are of the form

$$\underline{x}(t) = \underline{x}_H(t) + \underline{x}_p(t)$$

where $\underline{x}_H$ is the general solution to the associated hom eqn and $\underline{x}_p$ is any solution to $\circledast$

SEC 5.2. THE EIGENVALUE METHOD FOR HOMOGENEOUS SYSTEMS

To solve a system of the form

$$\underline{x}' = \overset{\text{nxn matrix of constants}}{\underline{\underline{A}}}\,\underline{x} \qquad (HE)$$

we try the following solution:

$$\underline{x} = \underset{\text{constant vector}}{\underline{v}}\, e^{\lambda t}$$

Plugging this into (HE) gives

$$\lambda\,\underline{v}\,e^{\lambda t} = \underline{\underline{A}}\,\underline{v}\,e^{\lambda t}$$

$$\Rightarrow \quad \underline{\underline{A}}\,\underline{v} = \lambda\,\underline{v}$$

$$\Rightarrow \quad \underline{\underline{A}}\,\underline{v} = \lambda\,\overset{\text{unit matrix}}{\underline{\underline{\mathbb{1}}}}\,\underline{v}$$

$$\Rightarrow \quad \underbrace{(\underline{\underline{A}} - \lambda\underline{\underline{\mathbb{1}}})}_{\underline{\underline{M}}_\lambda}\,\underline{v} = \overset{\text{zero vector}}{\underline{0}} \qquad \circledast$$

If $\det(\underline{\underline{M}}_\lambda) \neq 0$ we can multiply both sides by $\underline{\underline{M}}_\lambda^{-1}$ to get $\quad \underline{v} = \underline{\underline{M}}_\lambda^{-1}\,\underline{0} = \underline{0}$

which is not an interesting solution.

If $\det(\underline{\underline{M}}_\lambda) = 0$, there exists no $\underline{\underline{M}}_\lambda^{-1}$ and we will have multiple non $\underline{0}$ solutions for $\underline{v}$.

The method for finding λ's and nonzero $\underline{v}$'s that solve $\circledast$ is called the eigenvalue method.

EIGENVALUE METHOD

(1) First we find all possible λ such that $\det(\underline{\underline{M}}_\lambda)=0$. Since $\det(\underline{\underline{M}}_\lambda)$ is a polynomial of the n^{th} degree, we expect to find n solutions where some of the λ's might be equal.

(2) Next we find the possible $\underline{v}$'s. To do so, solve the equation $\underline{\underline{M}}_\lambda \underline{v} = \underline{0}$ for all values of λ. Given a λ, then the LI solutions for $\underline{v}$ are denoted by $\underline{v}_\lambda^{(1)}, \underline{v}_\lambda^{(2)}, \ldots, \underline{v}_\lambda^{(k)}$.

DEFINITION The values of λ for which $\underline{\underline{M}}_\lambda = \underline{\underline{A}} - \lambda \underline{\underline{1}}$ has $\det(\underline{\underline{M}}_\lambda) = 0$ are called eigenvalues of $\underline{\underline{A}}$. The vectors $\underline{v}_\lambda^{(1)}, \ldots, \underline{v}_\lambda^{(k)}$ for which $\underline{\underline{M}}_\lambda \underline{v}_\lambda^{(i)} = \underline{0}$ are called the eigenvectors of $\underline{\underline{A}}$ belonging to (or associated to) λ.

The multiplicity of an eigenvalue λ (as a root of the polynomial $\det(M_\lambda)$) is called the algebraic multiplicity of λ. The number k of LI eivenvectors $\underline{v}_\lambda^{(1)}, \ldots, \underline{v}_\lambda^{(k)}$ belonging to λ is called its geometric multiplicity. The difference between the algebraic & geometric multiplicity of λ is called its defect.

PROPERTIES OF EIGENVALUES & VECTORS

(1) If $\underline{v}_\lambda^{(1)}, \ldots, \underline{v}_\lambda^{(k)}$ are eigenvectors belonging to λ then $\underline{v} = c_1 \underline{v}_\lambda^{(1)} + \cdots + c_k \underline{v}_\lambda^{(k)}$ is also an eigenvector belonging to λ. With other words: any linear combination of eigenvectors belonging to an eigenvalue is still an

eigenvector for that eigenvalue.

(2) Every λ has at least 1 eigenvector

(3) The defect of any eigenvalue is at least 0. With other words: the algebraic multiplicity of λ is always bigger than or equal to its geometric multiplicity.

<u>Note</u> (2) & (3) together imply that if λ has alg. mult. 1, it must have exactly 1 LI eigenvector $\underline{v}_\lambda$.

(4) Eigenvectors belonging to different eigenvalues are always LI. If an $n \times n$ matrix $\underline{\underline{A}}$ has n different eigenvalues then it has n LI eigenvectors.

For this section we will assume that $\underline{\underline{A}}$ has n different eigenvalues. In that case we can solve

$$\underline{x}' = \underline{\underline{A}}\,\underline{x}$$

as follows:

(1) Find all eigenvalues $\lambda_1, \dots, \lambda_n$ and corresponding eigenvectors $\underline{v}_{\lambda_1}, \dots, \underline{v}_{\lambda_n}$ for $\underline{\underline{A}}$

(2) Construct n LI solutions as follows

$$\underline{x}_1 = \underline{v}_{\lambda_1} e^{\lambda_1 t}, \ \underline{x}_2 = \underline{v}_{\lambda_2} e^{\lambda_2 t}, \dots, \underline{x}_n = \underline{v}_{\lambda_n} e^{\lambda_n t}$$

(3) If all λ_i are real then

$$\underline{x} = \sum_{i=1}^{n} C_i \, \underline{v}_{\lambda_i} e^{\lambda_i t}$$

next lecture (If some λ_i are complex, we group them in complex pairs and create 2 LI real solutions from each pair.

EXAMPLES (1) Find a general solution to

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(1) Find e-vals of "$\underline{\underline{A}}$":

$$\det(\underline{\underline{M}}_\lambda) = 0$$

$$\Rightarrow \det(\underline{\underline{A}} - \lambda \underline{\underline{\mathbb{1}}}) = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow -(4-\lambda)(1+\lambda) - 6 = 0$$

$$\Rightarrow -\lambda^2 + 3\lambda + 10 = 0 \Rightarrow$$

$$\lambda_1 = \frac{-3+\sqrt{49}}{-2} = -2$$

$$\lambda_2 = \frac{-3-\sqrt{49}}{-2} = 5$$

(2) Find e-vecs for each λ

λ_1 Solve $\underline{\underline{M}}_{\lambda_1} \underline{V}_{\lambda_1} = \underline{0}$

$$\Rightarrow (\underline{\underline{A}} - \lambda_1 \underline{\underline{\mathbb{1}}}) \underline{V}_{\lambda_1} = \underline{0}$$

$$\Rightarrow \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 6\alpha + 2\beta = 0 \\ 3\alpha + \beta = 0 \end{cases}$$

Note that the 2nd equation is actually the same as the first one. This is no coincidence! Demanding that $\det(\underline{\underline{M}}_{\lambda})=0$ is the same as demanding that at least one of the rows of $\underline{\underline{M}}_{\lambda}$ is a linear combination of the other ones.

We find that $\beta = -3\alpha$ so $\underline{v}_{\lambda_1} = \begin{bmatrix} \alpha \\ -3\alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, where α can take any value.

Since these vectors are linearly dependent we can only get 1 solution from them and we can set α to any non-0 value, say $\alpha = 1$.

So $\underline{v}_{\lambda_1} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$\underline{\lambda_2}$ > We need to solve

$$\begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $\alpha = 2\beta$ and $\underline{v}_{\lambda_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

The final solution is then

$$\underline{x} = c_1 \underline{x_1} + c_2 \underline{x_2}$$
$$= c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t}$$

SEC 5.2. & 5.5. COMPLEX E-VALS & E VALS WITH DEFECTS

Last lecture: To solve $\overset{\substack{n\times 1 \text{ vector} \\ \downarrow}}{\underline{x}}' = \overset{\substack{\text{const} \\ \downarrow}}{\underline{\underline{A}}}\,\underline{x}$, we substitut
$\underline{x} = \underline{v}e^{\lambda t}$ and get an eigenvalue equation

$$(\underline{\underline{A}} - \underline{\underline{\mathbb{1}}}\lambda)\underline{v} = \underline{0}$$

To solve:

① Find e-vals λ by solving $\det(\underline{\underline{A}} - \underline{\underline{\mathbb{1}}}\lambda) = 0$

② For each e-val, find e-vecs by solving

$$(\underline{\underline{A}} - \underline{\underline{\mathbb{1}}}\lambda)\underline{v} = \underline{0}$$

If we are lucky then $\underline{\underline{A}}$ has n different real e-vals $\lambda_1, \dots, \lambda_n$, each with a unique e-vec $\underline{v}_{\lambda_1}, \dots, \underline{v}_{\lambda_n}$.
The solution is then

$$\underline{x} = \sum_{i=1}^{n} c_i \, \underline{v}_{\lambda_i} e^{\lambda_i t}$$

Sometimes, however, you might have

(a) some complex e-vals, and or

(b) defective e-vals

For now we will mainly focuss on Case (a) and case (b) where the algebraic multiplicity of any λ is at most 2 and its defect at most 1.

(a) COMPLEX EIGENVALUES

EXAMPLE Solve $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \underbrace{\begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}}_{\underline{\underline{A}}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(1) Find e-vals of $\underline{\underline{A}}$

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{\mathbb{1}}}) = 0 \Rightarrow \det\left(\begin{bmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow (4-\lambda)^2 + 9 = 0$$

$$\Rightarrow (4-\lambda)^2 = -9$$

$$\Rightarrow 4 - \lambda = \pm i 3$$

$$\Rightarrow \lambda_1 = 4 + 3i \qquad \lambda_2 = \overline{\lambda_1} = 4 - 3i$$

Lets ignore the fact that λ is complex and continue

② Find e-vecs per e-val

$\lambda_1 = 4 + 3i$

$$\begin{bmatrix} 4-(4+3i) & -3 \\ 3 & 4-(4+3i) \end{bmatrix} \overset{\underline{V}_{\lambda_1}}{\overset{\shortparallel}{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3i & -3 \\ 3 & -3i \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -3i\alpha - 3\beta = 0 & (E1) \\ 3\alpha - 3i\beta = 0 & (E2) \end{cases}$$

E1 is equivalent to E2 so we only need to solve

$$3\alpha - 3i\beta = 0 \Rightarrow \alpha = i\beta$$

So

$$\underline{v_{\lambda_1}} = \beta \begin{bmatrix} i \\ 1 \end{bmatrix} \quad \text{for all } \beta \in \mathbb{C}$$

Interestingly we don't need to know what $\underline{v_{\lambda_2}}$ is. If we look at the part of the solution that λ_1 provides, we see that

$$\underline{\tilde{x}} = c_1 \begin{bmatrix} i \\ 1 \end{bmatrix} e^{(4+3i)t} = c_1 \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{4t}(\cos(3t) + i\sin(3t))$$

$$= c_1 e^{4t} \left(\begin{bmatrix} -\sin(3t) \\ \cos(3t) \end{bmatrix} + i \begin{bmatrix} \cos(3t) \\ \sin(3t) \end{bmatrix} \right)$$

$$= c_1 \operatorname{Re}(\underline{\tilde{x}}) + c_1 \operatorname{Im}(\underline{\tilde{x}})\, i$$

But $\underline{\tilde{x}}$ solves $\underline{x}' = \underline{\underline{A}}\, \underline{x}$, i.e. $\tilde{x}' = \underline{\underline{A}}\, \tilde{x}$

or $$c_1 \operatorname{Re}(\underline{\tilde{x}})' + c_1 \operatorname{Im}(\underline{\tilde{x}})'\, i = \underline{\underline{A}}\, c_1 \operatorname{Re}(\underline{\tilde{x}}) + \underline{\underline{A}}\, c_1 \operatorname{Im}(\underline{\tilde{x}})\, i$$

$$\Rightarrow \begin{cases} \operatorname{Re}(\underline{\tilde{x}})' = \underline{\underline{A}} \operatorname{Re}(\underline{\tilde{x}}) \\ \operatorname{Im}(\underline{\tilde{x}})' = \underline{\underline{A}} \operatorname{Im}(\underline{\tilde{x}}) \end{cases}$$

so $\operatorname{Re}(\underline{\tilde{x}})$ and $\operatorname{Im}(\underline{\tilde{x}})$ each solve the equation.

Moreover, you can show that they are always linearly independent. Therefore

$$\underline{x} = c_1 \operatorname{Re}(\underline{\tilde{x}}) + c_2 \operatorname{Im}(\underline{\tilde{x}})$$

$$= c_1 e^{4t} \begin{bmatrix} -\sin(3t) \\ \cos(3t) \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} \cos(3t) \\ \sin(3t) \end{bmatrix}$$

General strategy for solving $\underline{x}' = \underline{\underline{A}}\,\underline{x}$ where A has n different (possibly complex) e-vals

① Calc e-vals of $\underline{\underline{A}}$: $\lambda_1, \dots, \lambda_n$

② For each real eval, calculate its evec, for each pair of complex conjugate evals, choose calculate the evec for 1 eval of each pair

③ Each tuple $(\lambda_i, \underline{v}_{\lambda_i})$ of a real e-vec and real e-val provides a term $c_i \underline{v_{\lambda_i}} e^{\lambda_i t}$ to the general solution

Each pair of complex conjugate evals $(\lambda_j, \bar{\lambda}_j)$ provides two terms

$$c_1 Re(\tilde{\underline{x}}) + c_2 Im(\check{\underline{x}})$$

to the general solution, where $\hat{\underline{x}} = \underline{v_{\lambda_j}} e^{\lambda_j t}$

(b) E-VALS WITH ALG MULT 2 & DEFECT 1

If there is a λ with algebraic mult 2, but defect 1 then we won't find n LI solutions to our system. In that case we can try the same trick as for LnDE's: Try $\underline{x} = (\underline{v_1} t + \underline{v_2}) e^{\lambda t}$ for that eigenvalue. Plugging this into $\underline{x}' = \underline{\underline{A}}\,\underline{x}$ gives

$$\lambda \cancel{e^{\lambda t}} (\underline{v_1} t + \underline{v_2}) + \underline{v_1} \cancel{e^{\lambda t}} = \underline{\underline{A}}\,\underline{v_1} t \cancel{e^{\lambda t}} + \underline{\underline{A}}\,\underline{v_2} \cancel{e^{\lambda t}}$$

$$\Rightarrow \lambda \underline{v_1} t + \lambda \underline{v_2} + \underline{v_1} = \underline{\underline{A}}\,\underline{v_1} t + \underline{\underline{A}}\, v_2$$

$$\Rightarrow 0 = (\underline{\underline{A}}\,\underline{v_1} - \lambda \underline{v_1}) t + \underline{\underline{A}}\,\underline{v_2} - \lambda \underline{v_2} - \underline{v_1} \qquad \text{for all } t \in \mathbb{R}$$

$$\Rightarrow \begin{cases} (\underline{\underline{A}} - \underline{\underline{\mathbb{1}}}\lambda)\, \underline{v_1} = \underline{0} \\ (\underline{\underline{A}} - \underline{\underline{\mathbb{1}}}\lambda)\, \underline{v_2} = \underline{v_1} \end{cases}$$

These equations are called generalized eigen vector equations.

EXAMPLE Solve $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\underline{\underline{A}}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

① Calculate e-vals of $\underline{\underline{A}}$

$$\det\left(\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}\right) = 0 \Rightarrow (1-\lambda)^2 = 0 \Rightarrow \lambda = 1$$

and alg mult $\lambda = 2$

② Calculate e-vecs for each λ

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \beta = 0 \Rightarrow \underline{v_{\lambda_1}} = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ forall } \alpha \in \mathbb{R}$$

So we only have 1 LI solution: $\underline{v_{\lambda_1}}\, e^{\lambda_1 t}$ ...

To get 2 LI solutions we solve

$$(\underline{\underline{A}} - \underline{\underline{\mathbb{1}}}\lambda)\, \underline{v_2} = \underline{v_1}$$

or $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \Rightarrow \delta = \alpha$

so $\underline{v_2} = \begin{bmatrix} \gamma \\ \alpha \end{bmatrix}$

and $\underline{x} = \left(\begin{bmatrix} \alpha \\ 0 \end{bmatrix} t + \begin{bmatrix} \gamma \\ \alpha \end{bmatrix} \right) e^t = \begin{bmatrix} \alpha t + \gamma \\ \alpha \end{bmatrix} e^t = \begin{bmatrix} c_1 t + c_2 \\ c_1 \end{bmatrix} e^t$

SEC 5.5. EIGENVALUES WITH DEFECTS

When solving $\underline{x}' = \underline{\underline{A}}\,x$ it might happen that an e-val λ of $\underline{\underline{A}}$ is defective, i.e., it does not provide enough e-vals to construct all solutions to $\underline{x}' = \underline{\underline{A}}\,\underline{x}$. If this is the case, we will need the theory of generalized e-vecs.

DEFINITIONS • If λ is an e-val of $\underline{\underline{A}}$ then then a rank r generalized e-vec is a vector $\underline{v}$, such that

$$(\underline{\underline{A}} - \lambda \underline{\underline{\mathbb{1}}})^{r-1}\underline{v} \neq \underline{0}$$

but

$$(\underline{\underline{A}} - \lambda \underline{\underline{\mathbb{1}}})^{r}\underline{v} = \underline{0}$$

- A length k-chain of generalized e-vecs based on the eigen vector $\underline{v}$ is a set $\{\underline{v}^{(1)}, \underline{v}^{(2)}, \dots, \underline{v}^{(k)}\}$ of generalized eigenvectors such that

$$(\underline{\underline{A}} - \lambda \underline{\underline{\mathbb{1}}})\underline{v}^{(k)} = \underline{v}^{(k-1)}$$

$$(\underline{\underline{A}} - \lambda \underline{\underline{\mathbb{1}}})\underline{v}^{(k-1)} = \underline{v}^{(k-2)}$$

$$\vdots$$

$$(\underline{\underline{A}} - \lambda \underline{\underline{\mathbb{1}}})\underline{v}^{(2)} = \underline{v}^{(1)}$$

$$\left[\ (\underline{\underline{A}} - \lambda \underline{\underline{\mathbb{1}}})\underline{v}^{(1)} = 0\ \right]$$

Notes • A generalized e-vec of rank 1 satisfies

$$\underbrace{(\underline{\underline{A}} - \lambda \underline{\underline{1}})^{0}}_{= \underline{\underline{1}}} \underline{v} \neq \underline{0} \quad \text{and} \quad (\underline{\underline{A}} - \lambda \underline{\underline{1}}) \underline{v} = \underline{0}$$

So its just a standard e-vec.

- For a k-chain $\{\underline{v}^{(1)}, \underline{v}^{(2)}, \dots, \underline{v}^{(k)}\}$ we have that

$$(\underline{\underline{A}} - \underline{\underline{1}}\lambda)^{i} \underline{v}^{(i)} = \underline{0} \qquad i = 1, \dots, k$$

- Given an e-vec $\underline{v}_1$ for a defect λ, you determine the length of the k-chain by solving $(\underline{\underline{A}} - \underline{\underline{1}}\lambda)\underline{v}^{(i)} = \underline{v}^{(i-1)}$ for increasing i until you have an unsolvable equation, in which case you have an $i-1$ chain
- An e-val λ can be defective and still have multiple e-vecs. Each of these e-vecs has its own chain of generalized e-vecs (where some might have length 1)

THEOREMS (1) Any matrix has n LI generalized eigenvectors.

(2) For the system $\underline{x}' = \underline{\underline{A}}\, \underline{x}$, any eigenvalue λ with chains of generalized e-vecs $\{\underline{v}_1^{(1)}, \dots, \underline{v}_1^{(k_1)}\}, \dots,$ $\{\underline{v}_l^{(1)}, \dots \underline{v}_l^{(k_l)}\}$ provides the following LI solutions per chain:

ith chain with length j

$$\begin{cases} \underline{x}_i^{(1)} = \underline{v}_i^{(1)} e^{\lambda t} \\ x_i^{(2)} = (\underline{v}_i^{(1)} t + v_i^{(2)}) e^{\lambda t} \\ x_i^{(3)} = (\underline{v}_i^{(1)} \frac{t^2}{2} + v_i^{(2)} t + v_i^{(3)}) e^{\lambda t} \\ \vdots \\ x_i^{(j)} = \left(\sum_{m=1}^{j} \frac{v_i^{(m)} t^{j-m}}{(j-m)!} \right) e^{\lambda t} \end{cases}$$

EXAMPLE Find all solutions to $\underline{x}' = \underline{\underline{A}}\,\underline{x}$ where

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

① Find all e-vals

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{1}}) = \det\left(\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^3$$

so $\lambda = 1$ with alg mult 3

② Find all chains of generalized e-vecs

let $\underline{\underline{M}} = (\underline{\underline{A}} - \lambda \underline{\underline{1}}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

then

$$\underline{\underline{M}}\,\underline{v} = \underline{0} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \gamma = 0$$

So $\underline{v} = \begin{bmatrix} \alpha \\ \beta \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\underline{\underline{A}}$ has 2 LI e-vecs: $\underline{v_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\underline{v_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow$ geomult$(\lambda) = 2$.

$\Rightarrow$ λ has defect 1 : we're missing one e-vec.

$\Rightarrow$ we need generalized e-vecs to solve this problem

Lets create chains of generalized e-vecs:

for $\underline{v}_1^{(1)} = \underline{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ we find that the equation

$\underline{\underline{M}}\,\underline{v}_1^{(2)} = \underline{v}_1^{(1)}$ leads to $\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$

which is unsolvable.

For $\underline{v}_2^{(1)} = \underline{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$

$$\underline{\underline{M}}\,\underline{v}_2^{(2)} = \underline{v}_2^{(1)} \Rightarrow \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix} \Rightarrow \gamma = 1$$

So any vector of the form $\underline{v}_2^{(2)} = \begin{bmatrix}\alpha\\\beta\\1\end{bmatrix}$ is a generalized evec for $\underline{v}_2$

Note that any choice of α, β gives a valid $\underline{v}_2^{(2)}$. Indeed since $\underline{v}_2^{(2)} = \alpha\,\underline{v}_1 + \beta\,\underline{v}_2 + \begin{bmatrix}0\\0\\1\end{bmatrix}$, the choice has no real effect once you plug it into the solution so you might as well set $\alpha = \beta = 0$.

All together we have that

$\{\underline{v}_1^{(1)}\} = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$ (chain 1)

$\lambda = 1$, has 2 chains:

$\{\underline{v}_2^{(1)}, \underline{v}_2^{(2)}\} = \left\{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ (chain 2)

Each provides the following solutions

$\underline{x}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}e^t$ (chain 1)

$$\underline{x}_2^{(1)} = \begin{bmatrix}0\\1\\0\end{bmatrix}e^t, \quad \underline{x}_2^{(2)} = \left(\begin{bmatrix}0\\1\\0\end{bmatrix}t + \begin{bmatrix}0\\0\\1\end{bmatrix}\right)e^t$$

so all together:

$$\underline{x} = c_1 \underline{x_1} + c_2 \underline{x_2}^{(1)} + c_3 \underline{x_2}^{(2)}$$

$$= \begin{bmatrix} c_1 e^t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ (c_2 + c_3 t) e^t \\ c_3 e^t \end{bmatrix}$$

$$= \begin{bmatrix} c_1 e^t \\ (c_2 + c_3 t) e^t \\ c_3 e^t \end{bmatrix}$$

What about complex e-vals?

For pairs of conjugate complex e-vals you choose 1 of the evals and set up the solutions provided by this e-val the same way as in the real case but at the end you split up your solutions into twice as many by using the real and imaginary parts.

REVISION MIDTERM 2

Sections 3.4 - 3.6, 4.1 - 4.2, 5.1 - 5.5

Original slides by Gert Vercleyen

IMPORTANT NOTE:

These slides do not revise all the material!
They're not sufficient to revise for the exam!

They're meant as a guide to some of the key concepts in the course.

CHAPTER 3

Mainly: Particular solutions to LnDEs and springs

Sec 3.5. Nonhomogeneous equations and undetermined coefficients

Non-hom equation

$$L(y) = \sum_{i=0}^{n} a_i D^i y = f(x)$$

general sol to hom eqn (with $f(x) = 0$)

Solution:

$$\boxed{y = y_h + y_p}$$

any 1 sol to whole eqn

Methods to find y_p:

1. **Undetermined Coefficients**
2. **Variation of parameters**

Sec 3.5. Nonhomogeneous equations and undetermined coefficients

Undetermined coefficients

1. **First of all**: if $f(x) = f_1(x) + \cdots + f_n(x)$ then $y_p = y_{p_1} + \cdots + y_{p_n}$, where $L(y_{p_i}) = f_i(x)$ for all i.

2. If $f(x)$ (or $f_i(x)$ if you split $f(x)$ up in parts) is of the form

$$p_m(x)e^{ax}\cos(bx) \quad \text{or} \quad p_m(x)e^{ax}\sin(bx)$$

m'th degree polynomial

There are 2 possibilities:

A. $f(x)$ nor any of its derivatives solve the hom eqn $L(y) = 0$

$$\Rightarrow y_p = (A_0 + A_1x + \cdots + A_mx^m)e^{ax}\cos(bx) + (B_0 + B_1x + \cdots + B_mx^m)e^{ax}\sin(bx)$$

B. $f(x)$ or any of its derivatives solves the hom eqn $L(y) = 0$

$$\Rightarrow y_p = x^s((A_0 + A_1x + \cdots + A_mx^m)e^{ax}\cos(bx) + (B_0 + B_1x + \cdots + B_mx^m)e^{ax}\sin(bx))$$

Where s is the smallest integer > 0 such that y_p does not contain a term that solve the hom eqn

Sec 3.5. Nonhomogeneous equations and undetermined coefficients

Variation of parameters

If $f(x)$ is not of the form

$$p_m(x)e^{ax}\cos(bx) \quad \text{or} \quad p_m(x)e^{ax}\sin(bx)$$

or some of the coefficients a_i are functions of x you need find y_p by using the following formula

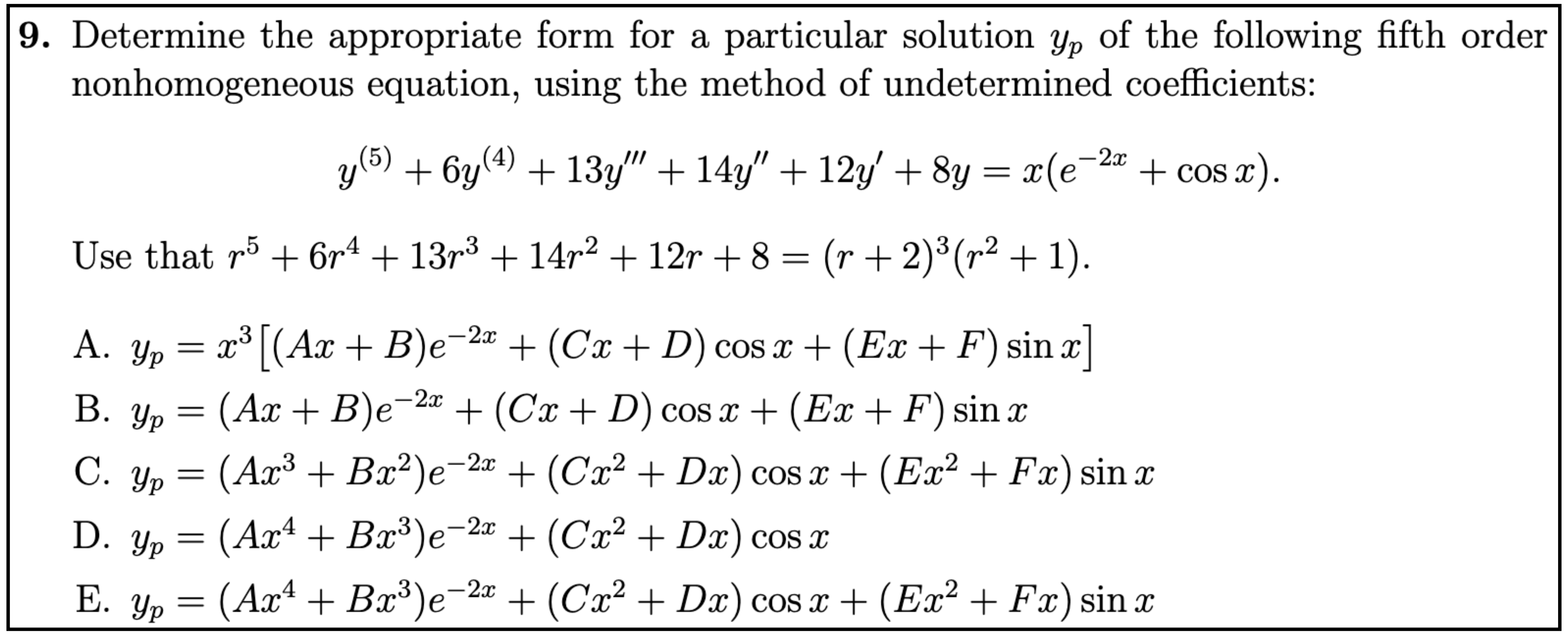
$$y_p = -y_1 \int \frac{y_2 f(x)}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 f(x)}{W(y_1, y_2)} dx$$

where y_1, y_2 are two linearly independent solutions to the hom eqn $L(y) = 0$ and $W(y_1, y_2)$ is their Wronskian

Sec 3.5. Nonhomogeneous equations and undetermined coefficients

TYPICAL QUESTIONS

Fall 2024 Final



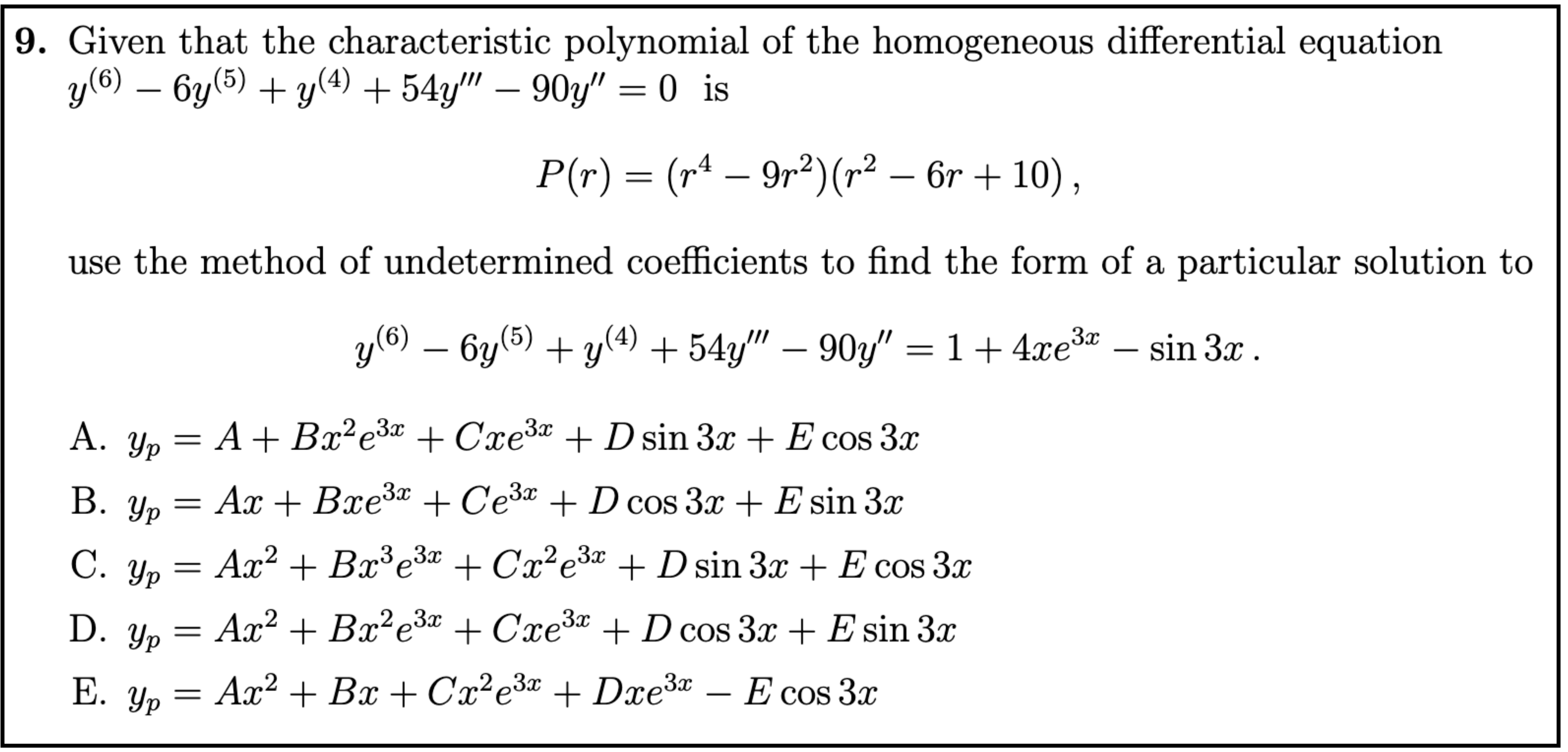
9. Determine the appropriate form for a particular solution y_p of the following fifth order nonhomogeneous equation, using the method of undetermined coefficients:

$$y^{(5)} + 6y^{(4)} + 13y''' + 14y'' + 12y' + 8y = x(e^{-2x} + \cos x).$$

Use that $r^5 + 6r^4 + 13r^3 + 14r^2 + 12r + 8 = (r+2)^3(r^2+1)$.

A. $y_p = x^3\big[(Ax+B)e^{-2x} + (Cx+D)\cos x + (Ex+F)\sin x\big]$
B. $y_p = (Ax+B)e^{-2x} + (Cx+D)\cos x + (Ex+F)\sin x$
C. $y_p = (Ax^3+Bx^2)e^{-2x} + (Cx^2+Dx)\cos x + (Ex^2+Fx)\sin x$
D. $y_p = (Ax^4+Bx^3)e^{-2x} + (Cx^2+Dx)\cos x$
E. $y_p = (Ax^4+Bx^3)e^{-2x} + (Cx^2+Dx)\cos x + (Ex^2+Fx)\sin x$

Spring 2024 Final



9. Given that the characteristic polynomial of the homogeneous differential equation $y^{(6)} - 6y^{(5)} + y^{(4)} + 54y''' - 90y'' = 0$ is

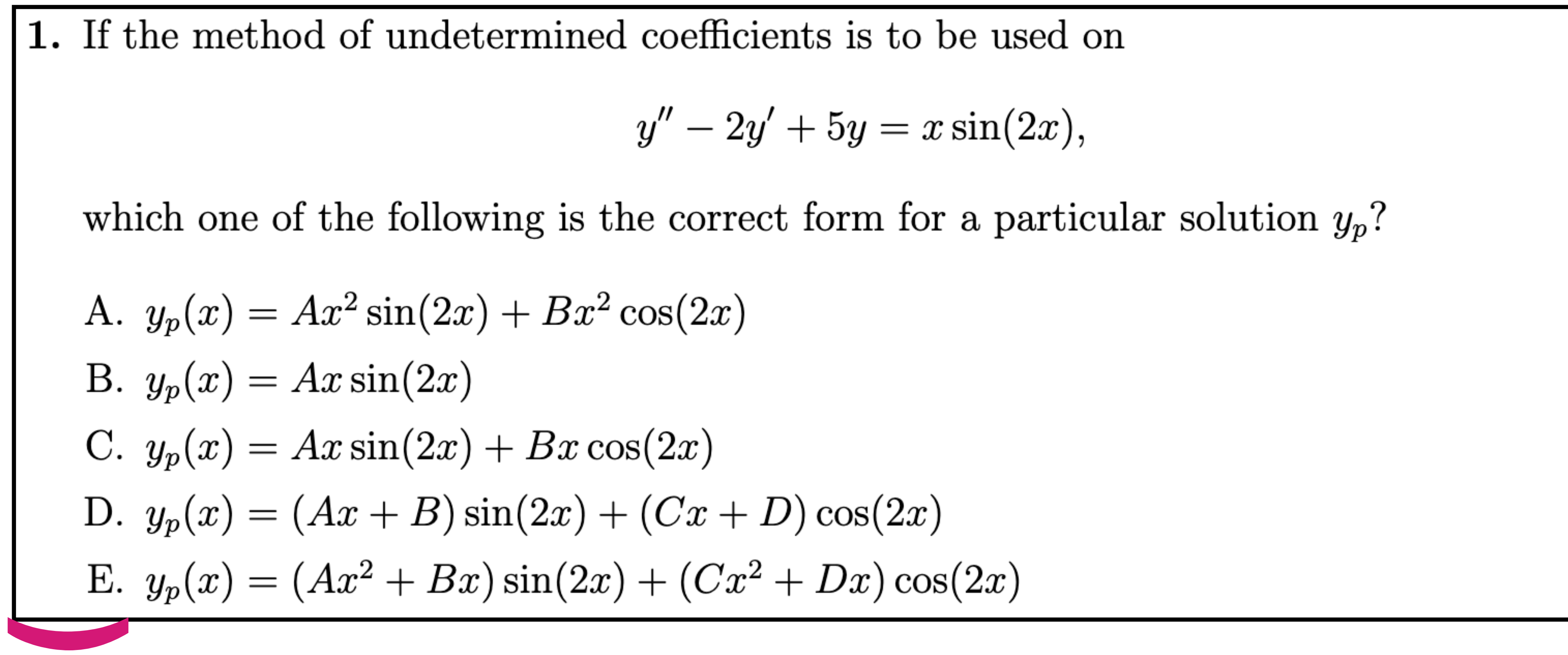
$$P(r) = (r^4 - 9r^2)(r^2 - 6r + 10)\,,$$

use the method of undetermined coefficients to find the form of a particular solution to

$$y^{(6)} - 6y^{(5)} + y^{(4)} + 54y''' - 90y'' = 1 + 4xe^{3x} - \sin 3x\,.$$

A. $y_p = A + Bx^2e^{3x} + Cxe^{3x} + D\sin 3x + E\cos 3x$
B. $y_p = Ax + Bxe^{3x} + Ce^{3x} + D\cos 3x + E\sin 3x$
C. $y_p = Ax^2 + Bx^3e^{3x} + Cx^2e^{3x} + D\sin 3x + E\cos 3x$
D. $y_p = Ax^2 + Bx^2e^{3x} + Cxe^{3x} + D\cos 3x + E\sin 3x$
E. $y_p = Ax^2 + Bx + Cx^2e^{3x} + Dxe^{3x} - E\cos 3x$

Forgot which exam :(



1. If the method of undetermined coefficients is to be used on

$$y'' - 2y' + 5y = x\sin(2x),$$

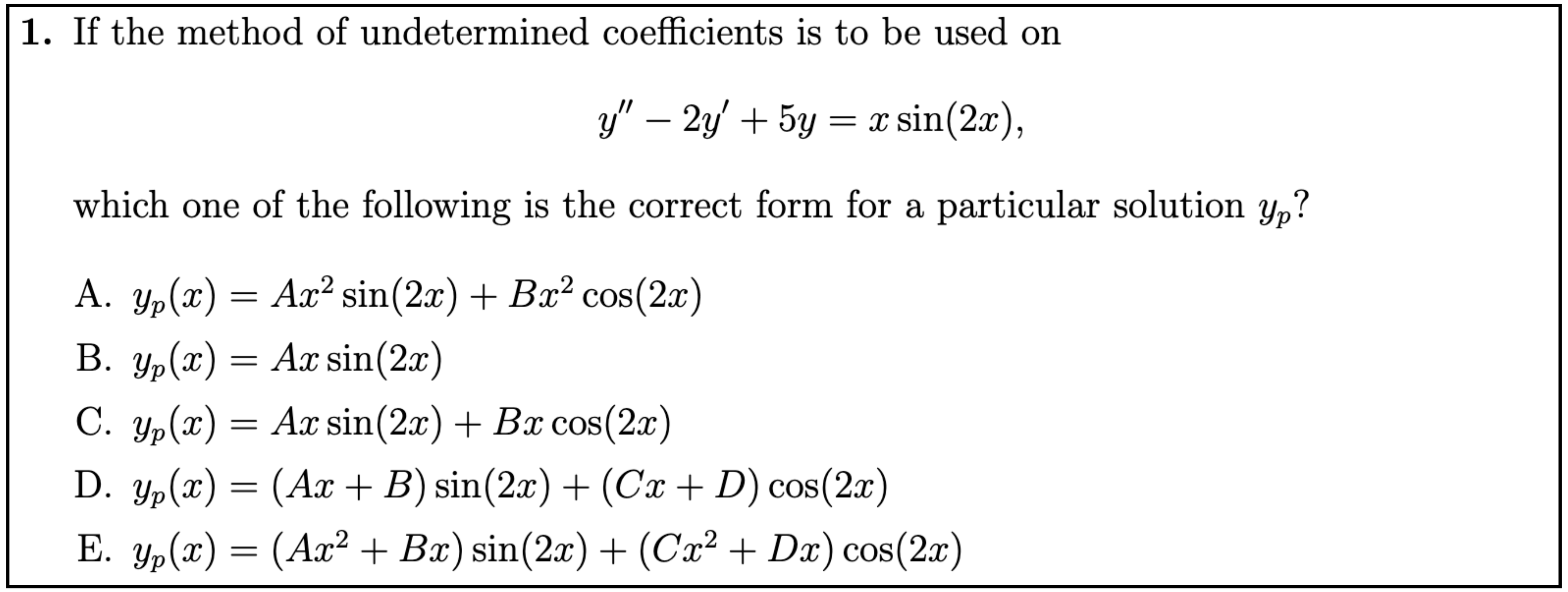
which one of the following is the correct form for a particular solution y_p?

A. $y_p(x) = Ax^2\sin(2x) + Bx^2\cos(2x)$
B. $y_p(x) = Ax\sin(2x)$
C. $y_p(x) = Ax\sin(2x) + Bx\cos(2x)$
D. $y_p(x) = (Ax+B)\sin(2x) + (Cx+D)\cos(2x)$
E. $y_p(x) = (Ax^2+Bx)\sin(2x) + (Cx^2+Dx)\cos(2x)$

Sec 3.5. Nonhomogeneous equations and undetermined coefficients

TYPICAL QUESTIONS

Spring 2024 Midterm 2



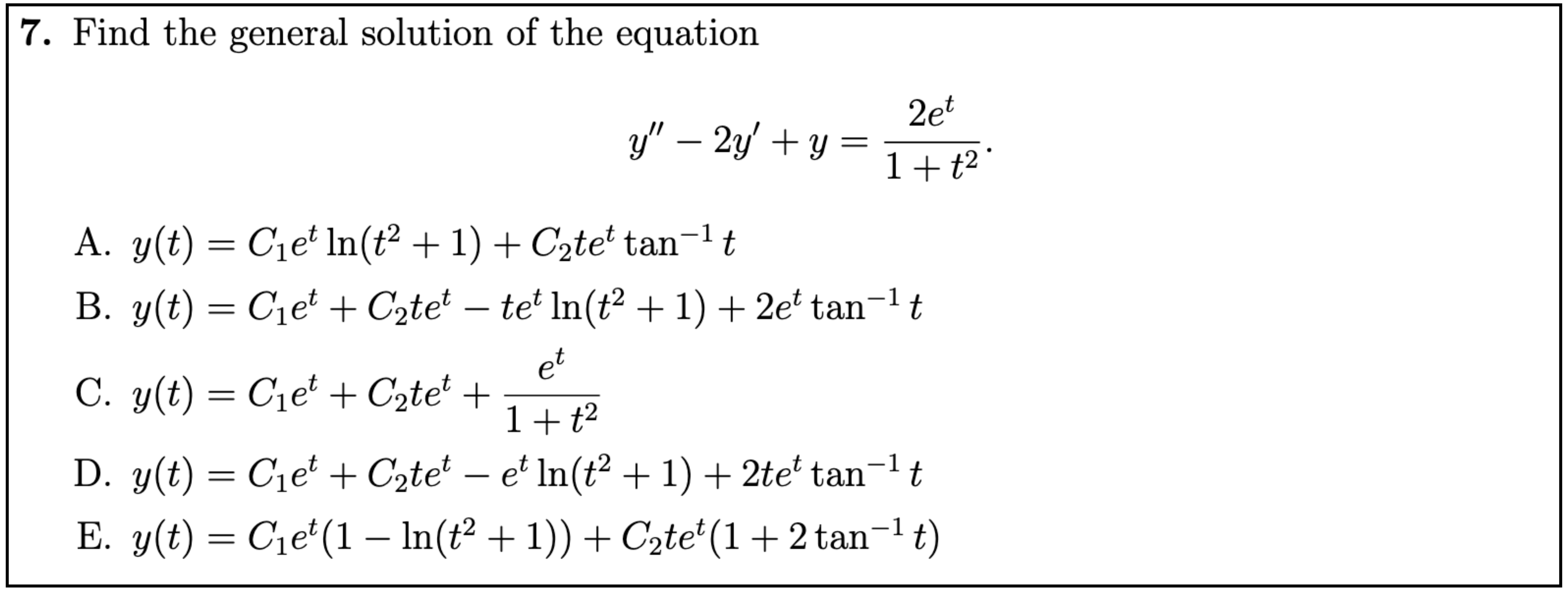
1. If the method of undetermined coefficients is to be used on

$$y'' - 2y' + 5y = x\sin(2x),$$

which one of the following is the correct form for a particular solution y_p?

A. $y_p(x) = Ax^2\sin(2x) + Bx^2\cos(2x)$
B. $y_p(x) = Ax\sin(2x)$
C. $y_p(x) = Ax\sin(2x) + Bx\cos(2x)$
D. $y_p(x) = (Ax + B)\sin(2x) + (Cx + D)\cos(2x)$
E. $y_p(x) = (Ax^2 + Bx)\sin(2x) + (Cx^2 + Dx)\cos(2x)$

Fall 2024 Final



7. Find the general solution of the equation

$$y'' - 2y' + y = \frac{2e^t}{1+t^2}.$$

A. $y(t) = C_1e^t\ln(t^2+1) + C_2te^t\tan^{-1}t$
B. $y(t) = C_1e^t + C_2te^t - te^t\ln(t^2+1) + 2e^t\tan^{-1}t$
C. $y(t) = C_1e^t + C_2te^t + \dfrac{e^t}{1+t^2}$
D. $y(t) = C_1e^t + C_2te^t - e^t\ln(t^2+1) + 2te^t\tan^{-1}t$
E. $y(t) = C_1e^t(1 - \ln(t^2+1)) + C_2te^t(1 + 2\tan^{-1}t)$

Sec 3.4 and 3.6 Mechanical vibrations

General equation for mass attached to spring

$$mx'' + cx' + kx = F_0 \cos(\omega t)$$

You need to know how to solve this equation for the following scenarios

- $F_0 = 0$: no external force so just Lin hom 2nd order DE
  - If c = 0: no damping. Solutions can be written as $A\cos(\omega_0 t - \alpha)$. **You should know how to convert** $c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$ **to** $A\cos(\omega_0 t - \alpha)$**!**
  - if $c \neq 0$: damping so either under/over/critically-damped system depending on values of m, c, k. **You should know which scenario occurs for which values!**

Sec 3.4 and 3.6 Mechanical vibrations

General equation for mass attached to spring

$$mx'' + cx' + kx = F_0 \cos(\omega t)$$

You need to know how to solve this equation for the following scenarios

- $F_0 \neq 0$: no external force so just Lin hom 2nd order DE
  - If $c = 0$: no damping. Resonance occurs if $\omega = \omega_0 \quad (= \sqrt{k/m})$!
  - if $c \neq 0$: damping so could have practical resonance.

Sec 3.4 and 3.6 Mechanical vibrations

General equation for mass attached to spring

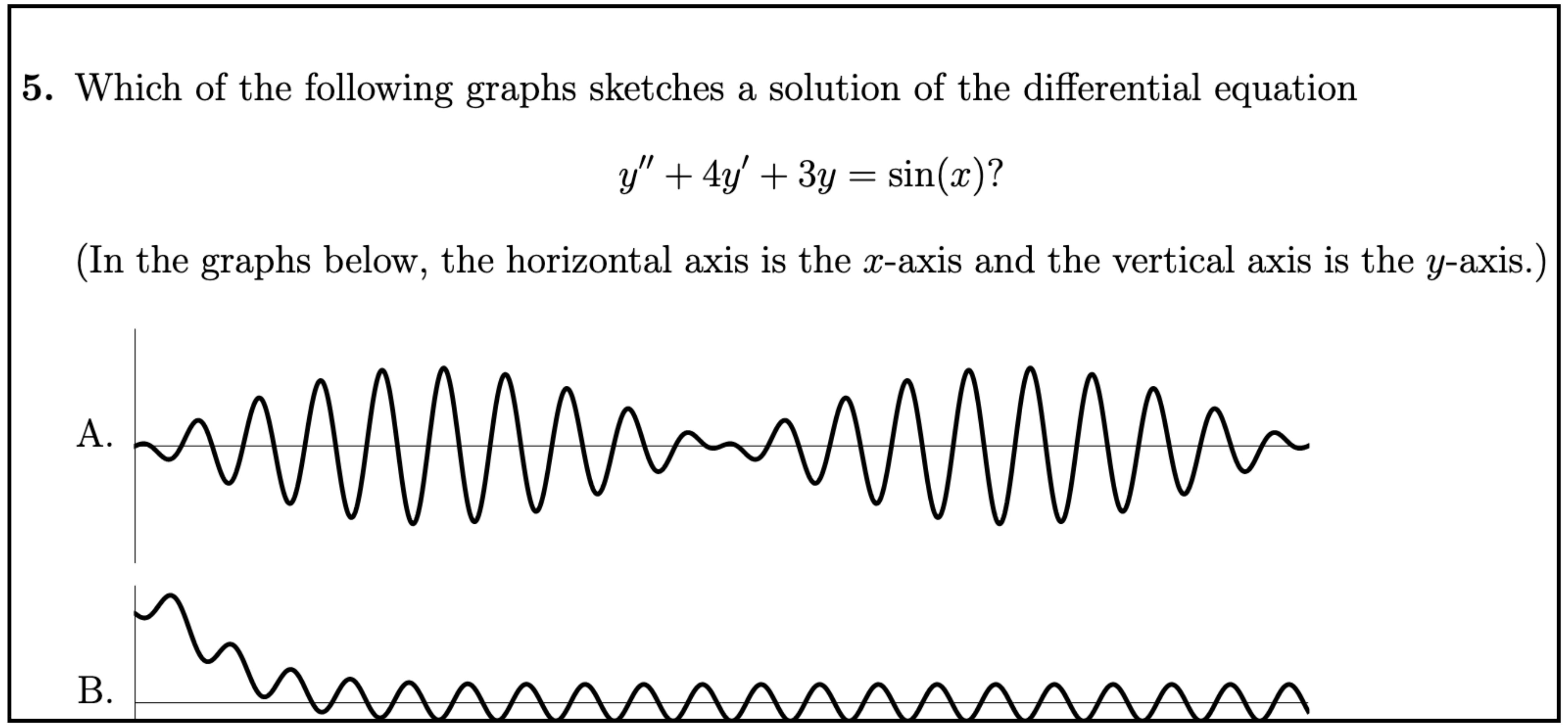
$$mx'' + cx' + kx = F_0 \cos(\omega t)$$

You should know the following terminology and how to obtain this data from the DE

- The natural angular velocity of the unforced spring: $\omega_0 = \sqrt{k/m}$
- The natural period of the motion of the unforced spring: $T = 2\pi/\omega_0$
- The natural frequency of the unforced spring: $f = \omega_0/(2\pi)$
- The amplitude of the motion of the unforced undamped spring: A.
  This depends on initial conditions: it is the value A in the solution
  $x = A\cos(\omega_0 t - \alpha)$

Sec 3.4 and 3.6 Mechanical vibrations

TYPICAL QUESTIONS (Fall 2024 Final)



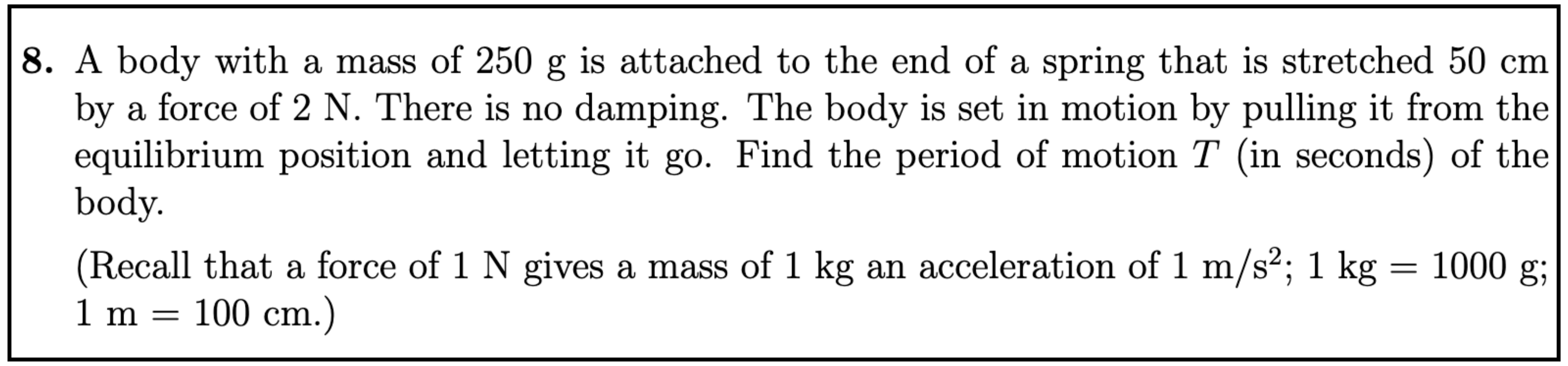
5. Which of the following graphs sketches a solution of the differential equation

$$y'' + 4y' + 3y = \sin(x)?$$

(In the graphs below, the horizontal axis is the x-axis and the vertical axis is the y-axis.)

A.

B.

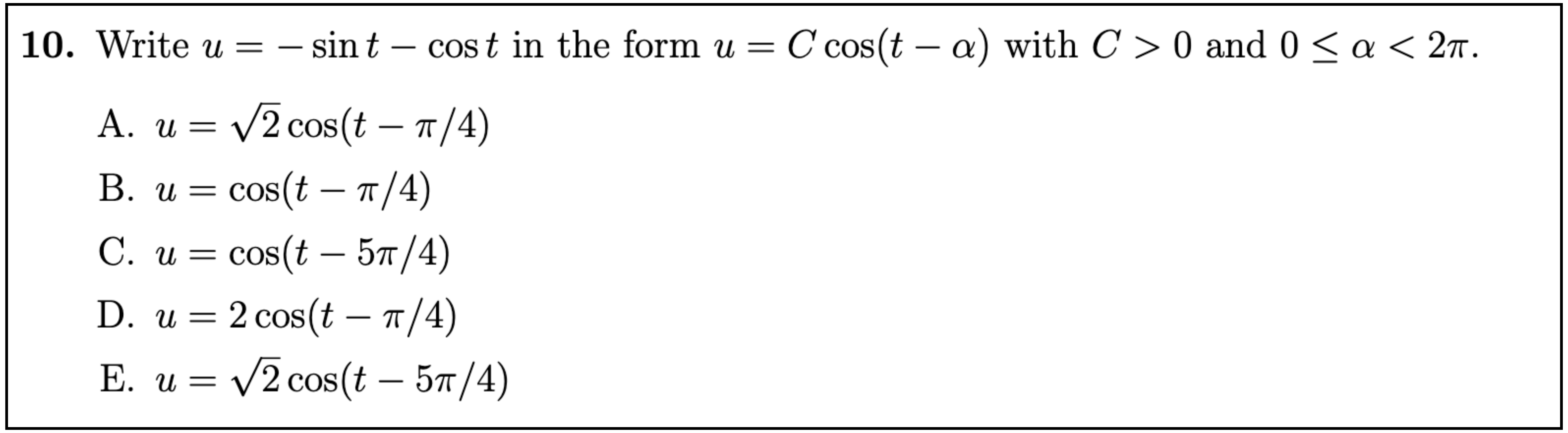


8. A body with a mass of 250 g is attached to the end of a spring that is stretched 50 cm by a force of 2 N. There is no damping. The body is set in motion by pulling it from the equilibrium position and letting it go. Find the period of motion T (in seconds) of the body.

(Recall that a force of 1 N gives a mass of 1 kg an acceleration of 1 m/s^2; 1 kg = 1000 g; 1 m = 100 cm.)

Sec 3.4 and 3.6 Mechanical vibrations

TYPICAL QUESTIONS (SPRING 2023 Final)



10. Write $u = -\sin t - \cos t$ in the form $u = C\cos(t - \alpha)$ with $C > 0$ and $0 \leq \alpha < 2\pi$.

A. $u = \sqrt{2}\cos(t - \pi/4)$

B. $u = \cos(t - \pi/4)$

C. $u = \cos(t - 5\pi/4)$

D. $u = 2\cos(t - \pi/4)$

E. $u = \sqrt{2}\cos(t - 5\pi/4)$

CHAPTER 4

Mainly: Mainly converting between systems of 1st order DEs and higher order DEs

Sec 4.1 First order systems and applications

CONVERTING FROM HIGHER ORDER DE TO SYSTEM

Equation $$x^{(n)} = f(x^{n-1}, \ldots, x', x, t)$$

Can be converted to system by setting $x_i = x^{(i)}$, so

$$\begin{cases} x_0' = x_1 \\ x_1' = x_2 \\ \vdots \\ x_{n-2}' = x_{n-1} \\ x_{n-1}' = f(x_{n-1}, \ldots, x_1, x_0, t) \end{cases}$$

Sec 4.1 First order systems and applications

STRONG THEOREM FOR SPECIAL LINEAR SYSTEMS

IVP of the form

$$\begin{cases} x_1' = \sum_{i=0}^{n} p_{1,i}(t)x_i + f_1(t) \\ \vdots \\ x_n' = \sum_{i=0}^{n} p_{n,i}(t)x_i + f_n(t) \\ x_i(a) = b_i \quad \text{for} \quad i = 1, \ldots n \end{cases}$$

(or equivalently, of the form $\mathbf{x}' = \mathbf{A}\mathbf{x}$), **satisfies: if all $p_{i,j}$ and f_i are continuous on some open interval I to which a belongs then the system has exactly 1 sol on I**

Sec 4.2 Method of elimination

CONVERTING FROM SYSTEM TO HIGHER ORDER

Elimination can be used to convert from system of eqns to (several) higher order equation(s). For details: see section 4.2 in the book.

Important remarks:

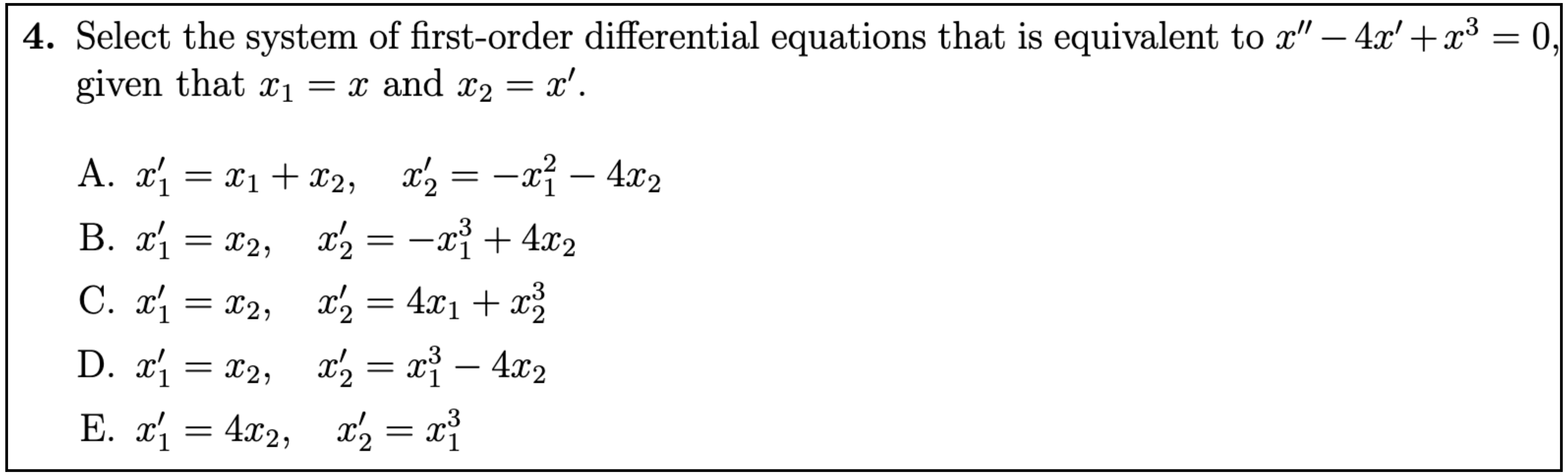
1. **Using elimination might create fake integration constants: constants that appear to be arbitrary but actually depend on one another. To know how many constants are expected: see pg 246 (green box) of the book**

2. **Because of this: avoid this technique whenever possible. If your system is in the form $\mathbf{x}' = \mathbf{A}\mathbf{x}$, always use techniques from chapter 5 to solve it! Techniques in this chapter are only useful for systems where the LHS of some equations contain multiple derivatives (e.g. eqns of the form like $x_1' + x_2' = \ldots, x_1' - 2x_2' = \ldots$)**

Sec 4.1-4.2

TYPICAL QUESTIONS

SPRING 2022 MIDTERM 2



4. Select the system of first-order differential equations that is equivalent to $x'' - 4x' + x^3 = 0$, given that $x_1 = x$ and $x_2 = x'$.

A. $x_1' = x_1 + x_2, \quad x_2' = -x_1^2 - 4x_2$

B. $x_1' = x_2, \quad x_2' = -x_1^3 + 4x_2$

C. $x_1' = x_2, \quad x_2' = 4x_1 + x_2^3$

D. $x_1' = x_2, \quad x_2' = x_1^3 - 4x_2$

E. $x_1' = 4x_2, \quad x_2' = x_1^3$

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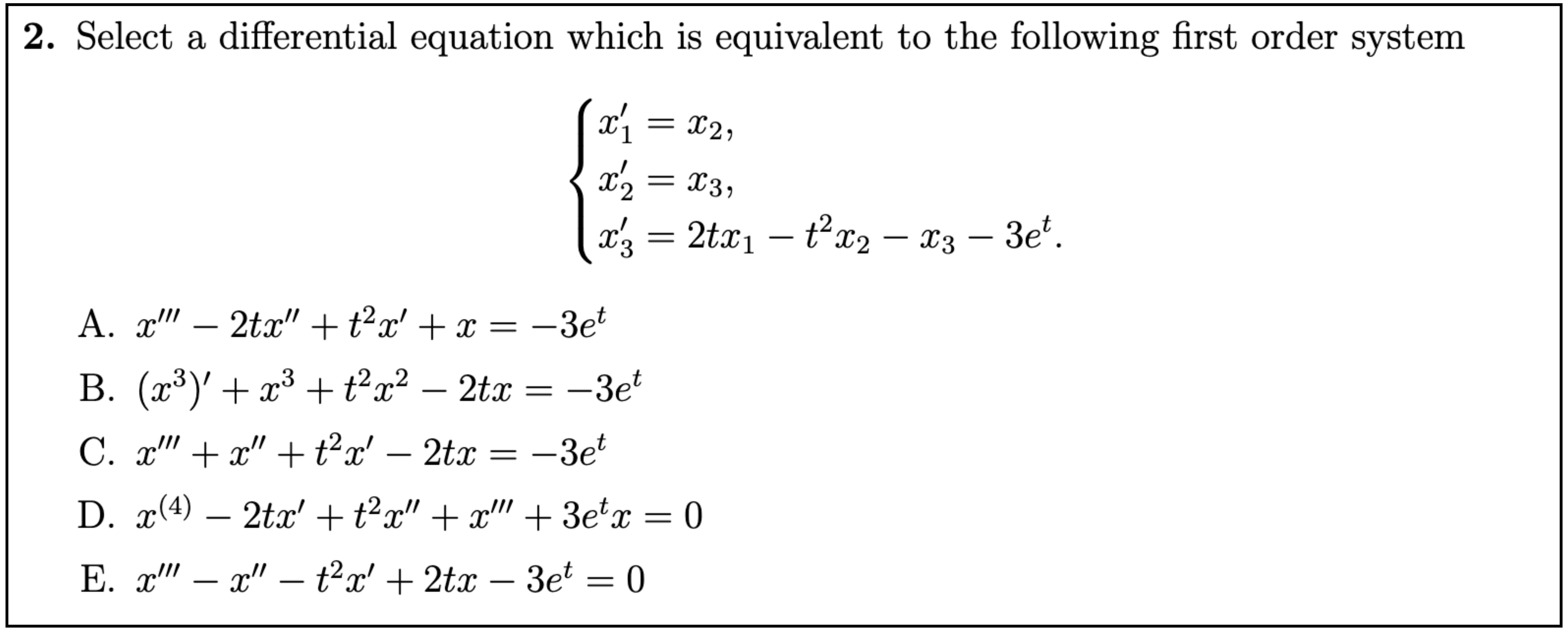

2. Select a differential equation which is equivalent to the following first order system

$$\begin{cases} x_1' = x_2, \\ x_2' = x_3, \\ x_3' = 2tx_1 - t^2x_2 - x_3 - 3e^t. \end{cases}$$

A. $x''' - 2tx'' + t^2x' + x = -3e^t$

B. $(x^3)' + x^3 + t^2x^2 - 2tx = -3e^t$

C. $x''' + x'' + t^2x' - 2tx = -3e^t$

D. $x^{(4)} - 2tx' + t^2x'' + x''' + 3e^tx = 0$

E. $x''' - x'' - t^2x' + 2tx - 3e^t = 0$

CHAPTER 5

Mainly: Solving systems of 1st order homogeneous DEs with constant coeffs

Sec 5.1, 5.2, 5.5

EQUATION

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(\mathbf{t})$$

Solution strategy

1. **Compute eigenvalues:** Let $\mathbf{M}_\lambda = \mathbf{A} - \lambda \mathbf{1}$, solve $\det(\mathbf{M}_\lambda) = 0$.
2. For each λ,
   1. Determine $k = \mathrm{algmult}(\lambda)$: how often does it appear as a root of the polynomial $\det(\mathbf{M}_\lambda)$?
   2. Find all eigenvectors for the eigenvalue λ by solving $\mathbf{M}_\lambda \mathbf{v} = \mathbf{0}$
   3. If you found k evecs, λ provides the following solutions: $\mathbf{x}_{\lambda,1} = \mathbf{v}_1 e^{\lambda t}, \ldots, \mathbf{x}_{\lambda,k} = \mathbf{v}_k e^{\lambda t}$

Sec 5.1, 5.2, 5.5

EQUATION

$$\mathbf{x}'(t) = \mathbf{Ax(t)}$$

Solution strategy

4. If you found less than k evecs: need to find generalized evecs of λ until we have k generalized evecs in total:
   To find gen evecs: for every evec $\mathbf{v}$ of λ compute chain $\{\mathbf{v}^{[1]}, \ldots, \mathbf{v}^{[m]}\}$ as follows.
   Start with $\mathbf{v}^{[1]} = \mathbf{v}$ and solve $\mathbf{M}_\lambda \mathbf{v}^{[i+1]} = \mathbf{v}^{[i]}$ with $i = 1$ to find $\mathbf{v}^{[2]}$, if this equation has a solution, solve $\mathbf{M}_\lambda \mathbf{v}^{[i+1]} = \mathbf{v}^{[i]}$ with $i = 2$ to find $\mathbf{v}^{[3]}$ and so on until the equation $\mathbf{M}_\lambda \mathbf{v}^{[i+1]} = \mathbf{v}^{[i]}$ has no solution.
   You construct a chain for every eigenvector of λ until you found a total (counting all evecs in all chains) of k evecs of λ.

Sec 5.1, 5.2, 5.5

EQUATION

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(\mathbf{t})$$

Solution strategy

5. Every chain $\{\mathbf{v}^{[1]}, \ldots, \mathbf{v}^{[m]}\}$ of evecs of λ provides the following solutions
$$\mathbf{x}_\lambda^{[1]} = \mathbf{v}^{[1]}e^{\lambda t}, \mathbf{x}_\lambda^{[2]} = (\mathbf{v}^{[1]}t + \mathbf{v}^{[2]})e^{\lambda t}, \mathbf{x}_\lambda^{[3]} = (\mathbf{v}^{[1]}\frac{t^2}{2} + \mathbf{v}^{[2]}t + \mathbf{v}^{[3]})e^{\lambda t}, \ldots$$

Note: If any of the λ are complex then that λ has a conjugate eval $\overline{\lambda}$ (which you should ignore) and it brings forth 2 solutions rather than 1.

These are just the real and complex part of the solution that λ provides:

$$\mathbf{x}_{\lambda,1} = \mathrm{Re}(\mathbf{v}_\lambda e^{\lambda t}), \mathbf{x}_{\lambda,2} = \mathrm{Im}(\mathbf{v}_\lambda e^{\lambda t})$$

If a complex eval is defect, i.e. there are not enough evecs to provide all solutions you just find generalized complex evecs and set the solution up as usual. At the end, you split every complex solution up in 2 real ones by taking real and imaginary parts.

Sec 5.1, 5.2, 5.5

EQUATION

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(\mathbf{t})$$

Solution strategy

5. Every chain $\{\mathbf{v}^{[1]}, \ldots, \mathbf{v}^{[m]}\}$ of evecs of λ provides the following solutions
$$\mathbf{x}_\lambda^{[1]} = \mathbf{v}^{[1]}e^{\lambda t}, \mathbf{x}_\lambda^{[2]} = (\mathbf{v}^{[1]}t + \mathbf{v}^{[2]})e^{\lambda t}, \mathbf{x}_\lambda^{[3]} = (\mathbf{v}^{[1]}\frac{t^2}{2} + \mathbf{v}^{[2]}t + \mathbf{v}^{[3]})e^{\lambda t}, \ldots$$

Note: If any of the λ are complex then that λ has a conjugate eval $\overline{\lambda}$ (which you should ignore) and it brings forth 2 solutions rather than 1.

These are just the real and complex part of the solution that λ provides:

$\mathbf{x}_{\lambda,1} = \mathrm{Re}(\mathbf{v}_\lambda e^{\lambda t}), \mathbf{x}_{\lambda,2} = \mathrm{Im}(\mathbf{v}_\lambda e^{\lambda t})$

If a complex eval is defect, i.e. there are not enough evecs to provide all solutions you just find generalized complex evecs and set the solution up as usual. At the end, you split every complex solution up in 2 real ones by taking real and imaginary parts.

Sec 5.1, 5.2, 5.5

TYPICAL QUESTIONS

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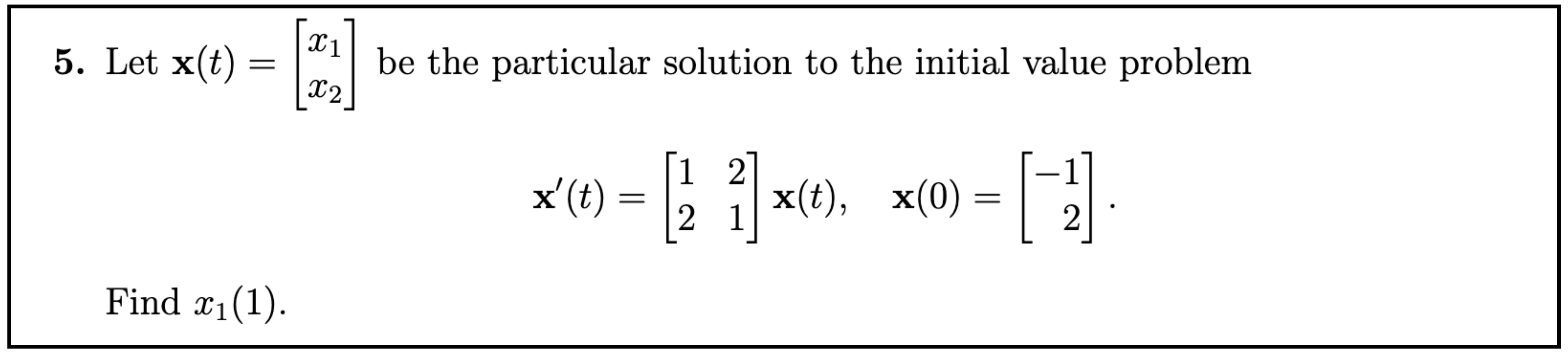
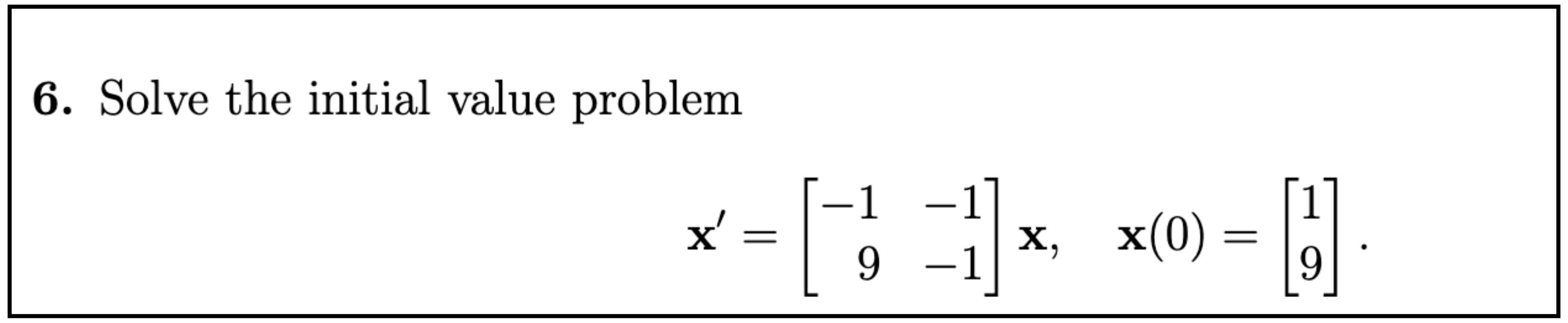


5. Let $\mathbf{x}(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the particular solution to the initial value problem

$$\mathbf{x}'(t) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{x}(t), \quad \mathbf{x}(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Find $x_1(1)$.



6. Solve the initial value problem

$$\mathbf{x}' = \begin{bmatrix} -1 & -1 \\ 9 & -1 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 9 \end{bmatrix}.$$

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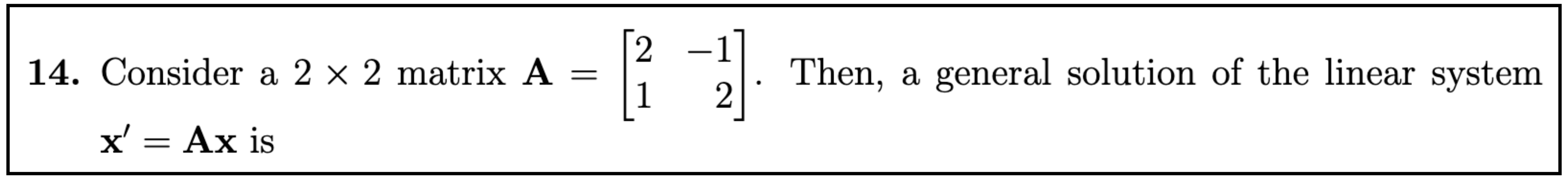
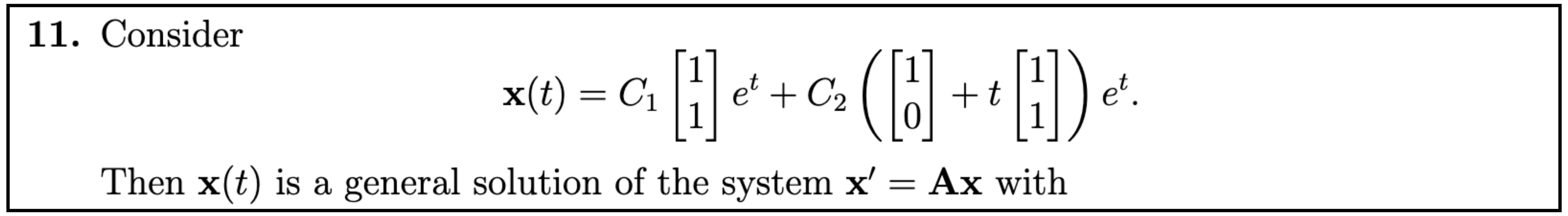


14. Consider a 2×2 matrix $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$. Then, a general solution of the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is



11. Consider

$$\mathbf{x}(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + C_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) e^t.$$

Then $\mathbf{x}(t)$ is a general solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with

Sec 5.3

For system with 2 equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$ you need to know what solutions look like based on the eigenvalues of the matrix $\mathbf{A}$.

An overview of all scenarios can be found in the book on pages 318-319.

It is important to know the names of the various situations!
NOTE: I forgot to mention the case of parallel lines with 2 zero eigenvalues and repeated eivenvector

Sec 5.3

TYPICAL QUESTIONS

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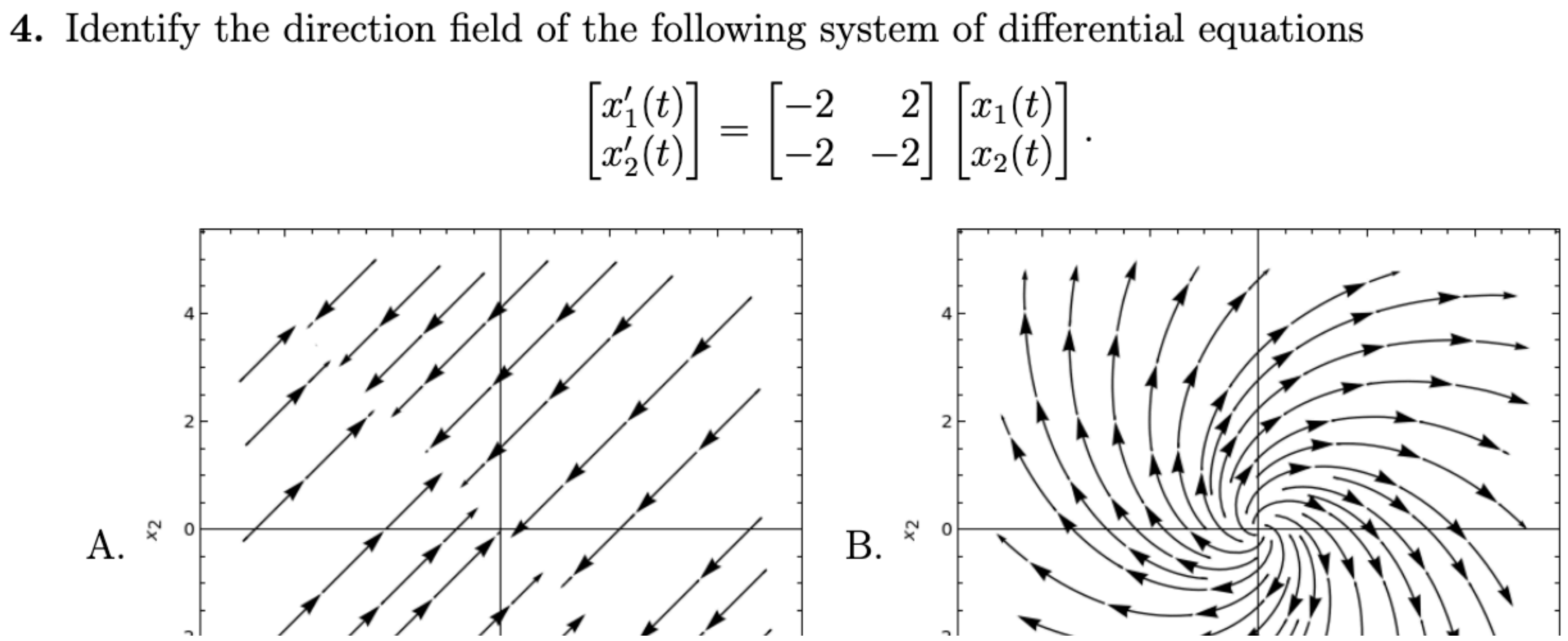

4. Identify the direction field of the following system of differential equations

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

A.

B.

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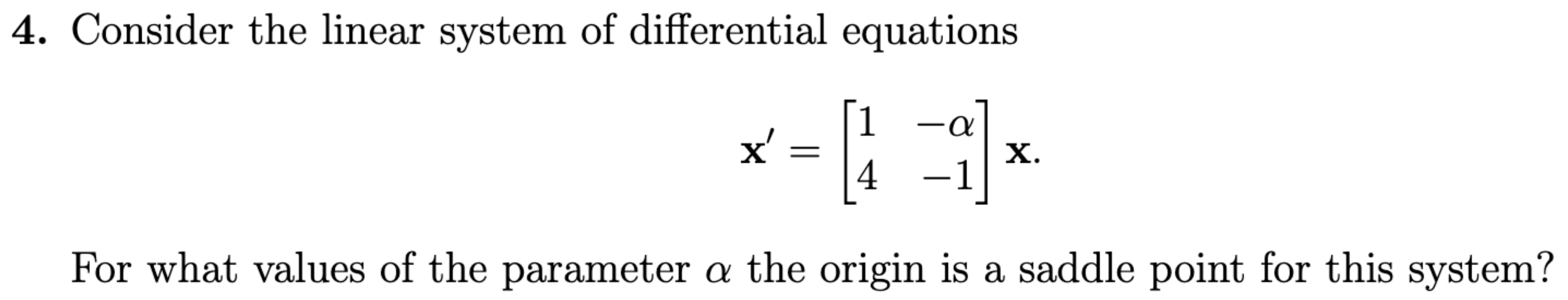
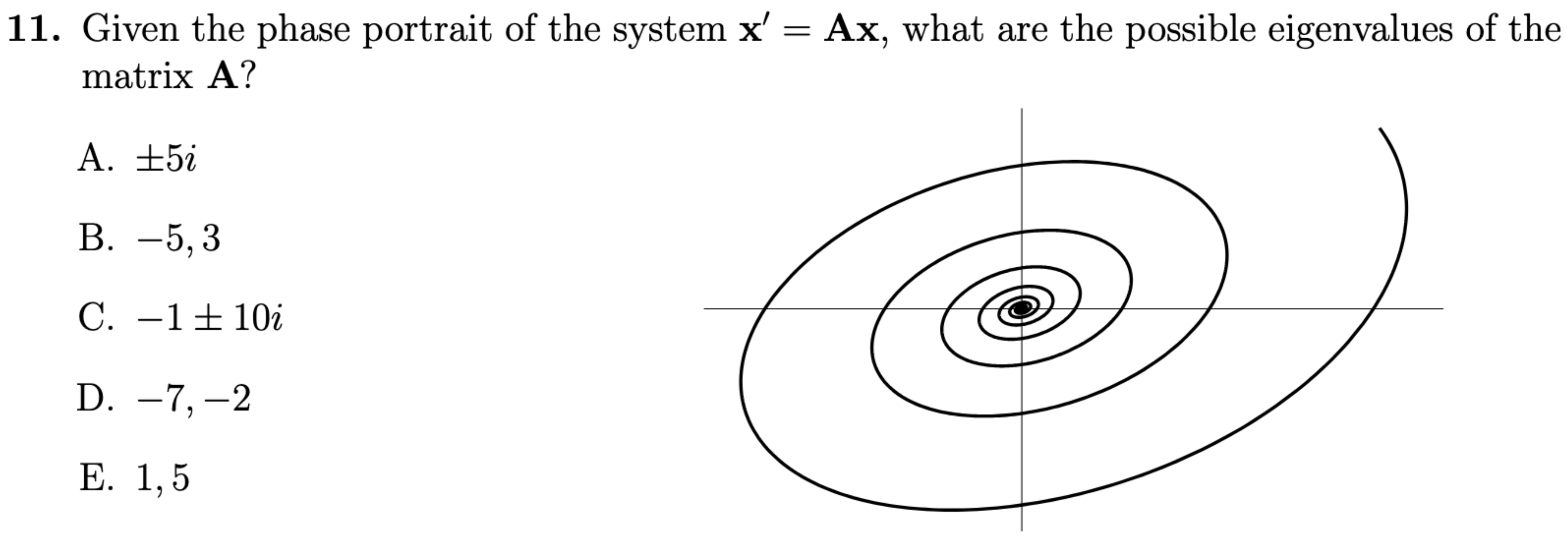


4. Consider the linear system of differential equations

$$\mathbf{x}' = \begin{bmatrix} 1 & -\alpha \\ 4 & -1 \end{bmatrix} \mathbf{x}.$$

For what values of the parameter α the origin is a saddle point for this system?

SPRING 2024 FINAL



11. Given the phase portrait of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, what are the possible eigenvalues of the matrix $\mathbf{A}$?

A. $\pm 5i$

B. $-5, 3$

C. $-1 \pm 10i$

D. $-7, -2$

E. $1, 5$

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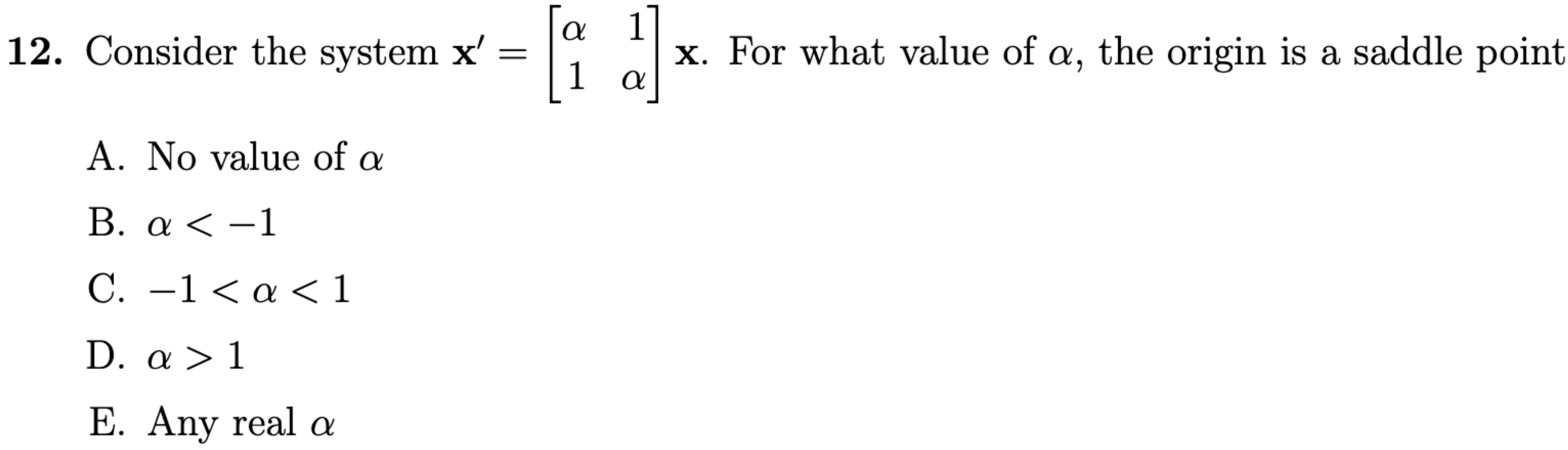


12. Consider the system $\mathbf{x}' = \begin{bmatrix} \alpha & 1 \\ 1 & \alpha \end{bmatrix} \mathbf{x}$. For what value of α, the origin is a saddle point

A. No value of α

B. $\alpha < -1$

C. $-1 < \alpha < 1$

D. $\alpha > 1$

E. Any real α

SEC 5.6 MATRIX EXPONENTIALS

Let

$$\underline{x}' = \underline{\underline{A}}\,\underline{x} \qquad (*)$$

with solution

$$\underline{x} = c_1 \underbrace{\underline{x}_1 + \cdots + c_n \underline{x}_n}_{n \times 1 \text{ matrices}}$$

Let

$$\underline{\underline{\Phi}}(t) = \left[\begin{array}{c:c:c:c} \underline{x}_1 & \underline{x}_2 & \cdots & \underline{x}_n \end{array}\right], \quad \underline{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

then we can also write $\underline{x}$ as

$$\underline{x} = \underline{\underline{\Phi}}(t)\,\underline{c}$$

Such $\underline{\underline{\Phi}}$ is called a *fundamental matrix* for $\circledast$

Properties

- $\det(\underline{\underline{\Phi}}) \neq 0$ since all $\underline{x}_i$ are LI. In particular: $\underline{\underline{\Phi}}$ is invertible.
- There can be many $\underline{\underline{\Phi}}$'s for the same system $\circledast$. They are, however, all related to each other via multiplication by invertible constant matrices: If $\underline{\underline{\tilde{\Phi}}}$ is also a fundamental matrix for $\circledast$ then there exists a constant matrix $\underline{\underline{M}}$ such that $\underline{\underline{\tilde{\Phi}}} = \underline{\underline{\Phi}}\,\underline{\underline{M}}$ for all t.

- If we're given an IVP

$$\begin{cases} \underline{x}' = \underline{\underline{A}}\,\underline{x} \\ \underline{x}(0) = \underline{x_0} \end{cases}$$

then, since $\underline{x} = \underline{\underline{\Phi}}(t)\,\underline{c} \Rightarrow \underline{x_0} = \underline{x}(0) = \underline{\underline{\Phi}}(0)\,\underline{c}$

$\Rightarrow \underline{c} = \left(\underline{\underline{\Phi}}(0)\right)^{-1}\underline{x_0}$, we have that

$$\underline{x} = \underline{\underline{\Phi}}(t)\left(\underline{\underline{\Phi}}(0)\right)^{-1}\underline{x_0}$$

To apply this formula, one needs to know how to invert matrices.

THEOREM Let $\underline{\underline{A}}$ be an invertible matrix, then

$$[\underline{\underline{A}}^{-1}]_{ij} = \frac{1}{\det(\underline{\underline{A}})}(-1)^{i+j}\det(\underline{\underline{C_{ji}}})$$

← Note that i,j are swapped

Where $\underline{\underline{C_{ji}}}$ is the matrix $\underline{\underline{A}}$ without its j^{th} row and i^{th} column.

EXAMPLES (1) If $\underline{\underline{A}} = \boxed{\begin{matrix} a & b \\ c & d \end{matrix}}$ is invertible then

$$\underline{\underline{A}}^{-1} = \frac{1}{\det(\underline{\underline{A}})}\boxed{\begin{matrix} +\det(C_{11}) & -\det(C_{21}) \\ -\det(C_{12}) & +\det(C_{22}) \end{matrix}}$$

with $C_{11} = \boxed{d}$, $C_{12} = \boxed{c}$ $C_{21} = \boxed{b}$, $C_{22} = \boxed{a}$

$$= \frac{1}{ad-bc}\boxed{\begin{matrix} d & -b \\ -c & a \end{matrix}}$$

(2) If $\underline{\underline{A}} = \boxed{\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}}$ is invertible then

$$\underline{\underline{A}}^{-1} = \frac{1}{\det(A)} \boxed{\begin{matrix} +\det C_{11} & -\det C_{21} & +\det C_{31} \\ -\det C_{12} & +\det C_{22} & -\det C_{32} \\ +\det C_{13} & -\det C_{23} & +\det C_{33} \end{matrix}}$$

with $C_{11} = \boxed{\begin{matrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{matrix}}$, $C_{12} = \boxed{\begin{matrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{matrix}}$, $C_{13} = \boxed{\begin{matrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}}$,

$\ldots$, $C_{33} = \boxed{\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}}$

So far all we did was introduce some extra notation. Now we will see a method that, in theory, allows us to compute $\underline{\underline{\Phi}}(t)$ directly from $\underline{\underline{A}}$.

Remember that for the 1st order DE $x' = ax$ we have the solution $x = e^{at} \cdot c$

If we look at the definition of e^{at}:

$$e^{at} := 1 + \frac{(at)^1}{1!} + \frac{(at)^2}{2!} + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(at)^n}{n!}$$

(where we set $(at)^0 = 1$ also for $t=0$)

This definition can be used to define $e^{\underline{\underline{A}}t}$:

$$e^{\underline{\underline{A}}t} = \sum_{n=0}^{\infty} \frac{(\underline{\underline{A}}t)^n}{n!}$$

(where we set $(\underline{\underline{A}}t)^0 = \underline{\underline{1}}$ also for $t=0$ or $\underline{\underline{A}} = \underline{\underline{0}}$)

THEOREM

① $e^{\underline{\underline{A}}t}$ exists for all $\underline{\underline{A}}$ and t

② $\frac{d}{dt}\left(e^{\underline{\underline{A}}t}\right) = \underline{\underline{A}}\, e^{\underline{\underline{A}}t}$

③ $e^{\underline{\underline{A}}t}$ is invertible for all $\underline{\underline{A}}$ and t, and $(e^{\underline{\underline{A}}t})^{-1} = e^{-\underline{\underline{A}}t}$

④ If $\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{B}}\,\underline{\underline{A}}$ then $e^{\underline{\underline{A}}} e^{\underline{\underline{B}}} = e^{\underline{\underline{A}}+\underline{\underline{B}}}$

(if $\underline{\underline{A}}\,\underline{\underline{B}} \neq \underline{\underline{B}}\,\underline{\underline{A}}$ this is <u>NOT</u> necessarilly true)

From ② it follows that $e^{\underline{\underline{A}}t}$ solves $\underline{x}' = \underline{\underline{A}}\,\underline{x}$. Moreover, if $\underline{x}(0) = \underline{x_0}$ then $\underline{x} = e^{\underline{\underline{A}}t}\,\underline{x_0}$.

In combination with $\underline{x} = \underline{\underline{\Phi}}(t)\left(\underline{\underline{\Phi}}(0)\right)^{-1}\underline{x_0}$

we see that

$$\underline{\underline{\Phi}}(t)\underbrace{\left(\underline{\underline{\Phi}}(0)\right)^{-1}}_{\text{const matrix}} = e^{\underline{\underline{A}}t}$$

So $e^{\underline{\underline{A}}t}$ is a fundamental matrix for $\underline{x}' = \underline{\underline{A}}\,\underline{x}$. You can also work in the other direction: if you know a fundamental solution $\underline{\underline{\Phi}}$ then you can obtain $e^{\underline{\underline{A}}t}$ as

$$e^{\underline{\underline{A}}t} = \underline{\underline{\Phi}}(t)\left(\underline{\underline{\Phi}}(0)\right)^{-1}$$

EXAMPLES

(1) Calculate $e^{\underline{\underline{A}}t}$ for $\underline{\underline{A}} = \boxed{\begin{matrix} a & 0 \\ 0 & b \end{matrix}}$.

In this case

$$e^{\underline{\underline{A}}t} = \sum_{n=0}^{\infty} \frac{\left(\boxed{\begin{matrix} at & 0 \\ 0 & bt \end{matrix}}\right)^n}{n!} = \sum_{n=0}^{\infty} \boxed{\begin{matrix} \frac{(at)^n}{n!} & 0 \\ 0 & \frac{(bt)^n}{n!} \end{matrix}}$$

$$= \boxed{\begin{matrix} \sum_{n=0}^{\infty} \frac{(at)^n}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{(bt)^n}{n!} \end{matrix}} = \boxed{\begin{matrix} e^{at} & 0 \\ 0 & e^{bt} \end{matrix}}$$

In general: for any diagonal matrix $\underline{\underline{A}} = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & a_n \end{bmatrix}$

$$e^{\underline{\underline{A}}t} = \begin{bmatrix} e^{a_1 t} & & & \\ & e^{a_2 t} & & \\ & & \ddots & \\ & & & e^{a_n t} \end{bmatrix}$$

Note: if $\underline{\underline{A}}$ can be written as $\underline{\underline{U}}\,\underline{\underline{D}}\,\underline{\underline{U}}^{-1}$ where D is diagonal then, since $(\underline{\underline{U}}\,\underline{\underline{D}}\,\underline{\underline{U}}^{-1})^n = \underline{\underline{U}}\,\underline{\underline{D}}^n\,\underline{\underline{U}}^{-1}$, we have that

$$e^{\underline{\underline{A}}t} = \underline{\underline{U}}\,\underbrace{e^{\underline{\underline{D}}t}}_{\text{easy to calculate.}}\,\underline{\underline{U}}^{-1}$$

If $\underline{\underline{A}}$ has no defect e-vals then it can be written as $\underline{\underline{U}}\,\underline{\underline{D}}\,\underline{\underline{U}}^{-1}$.

(2) Calculate $e^{\underline{\underline{A}}t}$ for $\underline{\underline{A}} = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}$.

Since $\underline{\underline{A}}^2 = \begin{bmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\underline{\underline{A}}^3 = \underline{\underline{0}}$, ... $\underline{\underline{A}}^i = \underline{\underline{0}}$ for $i \geq 3$.

$$e^{\underline{\underline{A}}t} = \underline{\underline{1}} + \underline{\underline{A}}t + \frac{\underline{\underline{A}}^2 t^2}{2} + \cancel{\frac{\underline{\underline{A}}^3 t^3}{6}} + \ldots$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & at & bt \\ 0 & 0 & ct \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{act^2}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & at & bt + \frac{ac}{2}t^2 \\ 0 & 1 & ct \\ 0 & 0 & 1 \end{bmatrix}$$

A matrix $\underline{\underline{A}}$ for which $\underline{\underline{A}}^n = \underline{\underline{0}}$ for some n is called nilpotent. For nilpotent matrices we only need to add n terms in the expansion of $e^{\underline{\underline{A}}t}$.

(3) Calculate $e^{\underline{\underline{A}}t}$ for $\underline{\underline{A}} = \boxed{\begin{matrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{matrix}}$

In this case $\underline{\underline{A}}$ is neither nilpotent nor diagonal. It can, however be written as

$$\underline{\underline{A}} = \overset{\overset{\text{diagonal}}{\downarrow}}{2\underline{\underline{\mathbb{1}}}} + \overset{\overset{\text{nilpotent}}{\swarrow}}{\underline{\underline{N}}}$$

and since $\underbrace{2\underline{\underline{\mathbb{1}}}\,\underline{\underline{N}} = \underline{\underline{N}}(2\underline{\underline{\mathbb{1}}})}_{\text{ALWAYS CHECK WHETHER THIS IS TRUE!}}$ $\quad e^{\underline{\underline{A}}t} = e^{(2\underline{\underline{\mathbb{1}}}+\underline{\underline{N}})t} = e^{2\underline{\underline{\mathbb{1}}}t}e^{\underline{\underline{N}}t}$

Since

$$e^{2\underline{\underline{\mathbb{1}}}t} = \boxed{\begin{matrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{matrix}}$$

and

$$e^{\underline{\underline{N}}t} = \boxed{\begin{matrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{matrix}}$$

$$e^{\underline{\underline{A}}t} = e^{2t}\boxed{\begin{matrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{matrix}}$$

Note: Every matrix $\underline{\underline{A}}$ can be written as $\underline{\underline{A}} = \underline{\underline{U}}^{-1}\underline{\underline{J}}\,\underline{\underline{U}}$ where $\underline{\underline{J}} = \underset{\underset{\text{diagonal}}{\uparrow}}{\underline{\underline{D}}} + \underset{\underset{\text{nilpotent}}{\uparrow}}{\underline{\underline{N}}}$ and $\underline{\underline{D}}\,\underline{\underline{N}} = \underline{\underline{N}}\,\underline{\underline{D}}$

This is called the Jordan decomposition of a matrix.

In particular we have that

$$e^{\underline{\underline{A}}t} = \underline{\underline{U}}^{-1} e^{(\underline{\underline{D}}+\underline{\underline{N}})t} U$$

$$= \underline{\underline{U}}^{-1} e^{\underline{\underline{D}}t} e^{\underline{\underline{N}}t} U$$

Since the jordan decomposition is not part of this curriculum we need to use the relation

$$e^{\underline{\underline{A}}t} = \underline{\underline{\Phi}}(t)\left(\Phi(0)\right)^{-1}$$

with $\underline{\underline{\Phi}}(t)$ any fundamental matrix for the system

$$\underline{x}' = \underline{\underline{A}}\,\underline{x}$$

5.7. NON HOMOGENEOUS LINEAR SYSTEMS

Now that we have all techniques to solve

$$\underline{x}' = \underline{\underline{A}}\,\underline{x} \qquad \text{(HE)}$$

we can turn to the question of solving

$$\underline{x}' = \underline{\underline{A}}\,\underline{x} + \underline{f}(t) \qquad \circledast$$

Just like for LnDE's we have that

$$\underline{x} = \underbrace{\underline{x_H}}_{\text{general sol to (HE)}} + \underbrace{\underline{x_p}}_{\text{any solution to } \circledast}$$

general sol to (HE) ↓ contains n integration constants

any solution to ⊛ ↓ contains no integration constants

In sec 3.5. we saw 2 techniques for finding x_p, and we can use a variant of each of them for finding $\underline{x_p}$ as well. Note that if $\underline{f}(t) = \underline{f_1}(t) + \dots + \underline{f_m}(t)$ then you can still split up the problem in finding $\underline{x_{p_1}}, \dots, \underline{x_{p_m}}$ and setting $\underline{x_p} = \underline{x_{p_1}} + \dots + \underline{x_{p_m}}$

TECHNIQUES FOR FINDING $\underline{x_p}$

1. UNDETERMINED COEFFICIENTS

If $\underline{f}(t)$ is of the form

$$\left(\sum_{i=0}^{m} \underline{v_i}\, t^{i}\right) e^{at} \cos(bt)$$

or

$$\left(\sum_{i=0}^{m} \underline{v_i}\, t^{i}\right) e^{at} \sin(bt)$$

then, if (lets assume its of form (1))

(A) None of the terms or their derivatives in (1) solve (HE):

$$\underline{x_p} = \left(\underline{c_0} + \underline{c_1} t + \dots + \underline{c_m} t^{m}\right) e^{\alpha t} \cos(\beta t) + \left(\underline{c_{m+1}} + \underline{c_{m+2}} t + \dots + \underline{c_{2m}} t^{m}\right) e^{\alpha t} \sin(\beta t)$$

where $\underline{c_0}, \dots, \underline{c_{2m}}$ are constant vectors whose values can be determined by plugging $\underline{x_p}$ into the DE ⊛

(B) One or more terms or their derivatives in (1) solve (HE) then

$$\underline{x_p} = \left[\underline{b_0} + \underline{b_1} t + \dots + \underline{b_{s-1}} t^{s-1} + t^{s}\left(\underline{c_0} + \underline{c_1} t + \dots + \underline{c_m} t^{m}\right)\right] e^{\alpha t} \cos(\beta t) + \left[\underline{b_s} + \underline{b_{s+1}} t + \dots + \underline{b_{2s-1}} t^{s-1} + t^{s}\left(\underline{c_{m+1}} + \underline{c_{m+2}} t + \dots + \underline{c_{2m}} t\right)\right] e^{\alpha t} \sin(\beta t)$$

THESE WERE NOT THERE FOR LnDEs BUT ARE NEEDED NOW !

Here $\underline{b_0}, \dots, \underline{b_{2s-1}}$ are also const vecs whose values are determined by plugging $\underline{x_p}$ into the equation ⊛

EXAMPLES (1) Find $\underline{x_p}$ for

$$\underline{x}' = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \underline{x} + \begin{bmatrix} 3 \\ 2t \end{bmatrix}$$

First we solve the hom eqn $\underline{x}' = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \underline{x}$

(1) Evals

$$(3-\lambda)(5-\lambda)-14=0 \Leftrightarrow 15-8\lambda+\lambda^2-14=0$$

$$\Leftrightarrow \lambda^2-8\lambda+1=0$$

$$\Leftrightarrow \lambda=\frac{8\pm\sqrt{32}}{2}$$

Since $\lambda=0$ is not a solution we can't get a polynomial $\underline{a}+\underline{b}\,t$ as a solution to the (HE)

↑ only appears if $\lambda=0$ with $\text{defect}(\lambda)=1$.

The method of undetermined coefficients says that in that case: $\underline{x_p}=\underline{a}+\underline{b}\,t.$

$$\underline{f}(t)=\begin{bmatrix}3\\2t\end{bmatrix}=\begin{bmatrix}3\\0\end{bmatrix}+\begin{bmatrix}0\\2\end{bmatrix}t$$

Lets try

$$\underline{x_p}=\underline{a}+\underline{b}\,t=\begin{bmatrix}a_1\\a_2\end{bmatrix}+\begin{bmatrix}b_1\\b_2\end{bmatrix}t=\begin{bmatrix}a_1+b_1t\\a_2+b_2t\end{bmatrix}$$

then

$$\underline{x_p}'=\begin{bmatrix}b_1\\b_2\end{bmatrix}$$

$$\underline{\underline{A}}\,\underline{x_p}+\underline{f}(t)=\begin{bmatrix}3(a_1+b_1t)+2(a_2+b_2t)\\7(a_1+b_1t)+5(a_2+b_2t)\end{bmatrix}+\begin{bmatrix}3\\2t\end{bmatrix}$$

$$=\begin{bmatrix}3a_1+2a_2+3+t(3b_1+2b_2)\\7a_1+5a_2+t(b_1+b_2+2)\end{bmatrix}$$

$$\text{So}\quad \begin{cases} b_1 = 3a_1 + 2a_2 + 3 \\ 3b_1 + 2b_2 = 0 \\ 7a_1 + 5a_2 = 0 \\ b_2 = b_1 + b_2 + 2 \end{cases} \Longrightarrow \begin{cases} b_1 = -2 \\ b_2 = 3 \\ 3a_1 + 2a_2 = 1 \\ 7a_1 + 5a_2 = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} b_1 = -2 \\ b_2 = 3 \\ a_2 = -1 \\ a_1 = \frac{5}{7} \end{cases} \Longrightarrow \boxed{\underline{x}_p = \begin{bmatrix} \frac{5}{7} \\ -1 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} t}$$

(2) Find a general form

(without determining the coefficients) for $\underline{x}_p$ when

$$\underline{x}' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \underline{x} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} t e^{-2t}$$

First we find the general form of the solution to $\underline{x}' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \underline{x}$

① <u>Evals</u>

$$(4-\lambda)(-1-\lambda) - 6 = 0$$

$$\Rightarrow -4 - 3\lambda + \lambda^2 - 6 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 10 = 0$$

$$\text{So}\quad \lambda_1 = \frac{3 + \sqrt{9+40}}{2} = 5$$

$$\lambda_2 = \frac{3-7}{2} = -2$$

$$\text{So}\quad \underline{x} = c_1 \underline{v_1} e^{5t} + c_2 \underline{v_2} e^{-2t}$$

Since the proposed form for $\underline{x_p}$

$$\underline{x_p} \overset{?}{=} (\underline{a_0} + \underline{a_1}\, t)\, e^{-2t} = \underline{\underline{a_0}\, e^{-2t}} + \underline{a_1}\, t\, e^{-2t}$$

contains a term that coincides with the solution to the homogeneous equation, we need to use the form proposed in case (B) of Method 1:

$$\underline{x_p} = \left(\underline{b_0} + \underline{b_1} t + \dots + \underline{b_{s-1}}\, t^{s-1} + t^{s}(\underline{a_0} + \underline{a_1} t)\right) e^{-2t}$$

where s is the smallest pos int such that $t^{s}(\underline{a_0} + \underline{a_1} t) e^{-2t}$ contains no term that solves the HE. In our case $s = 1$ so

$$\begin{aligned} \underline{x_p} &= (\underline{b_0} + t(\underline{a_0} + \underline{a_1} t))e^{-2t} \\ &= (\underline{b_0} + \underline{a_0} t + \underline{a_1} t^2)\, e^{-2t} \end{aligned}$$

2. VARIATION OF PARAMETERS

See the book if you want to know how to obtain the formula.
The formula is

$$\boxed{\underline{x_p} = \underline{\underline{\Phi}}(t) \int (\underline{\underline{\Phi}}(t))^{-1}\, \underline{f}(t)\, dt}$$

looks simple but is, in practice, very hard to use since:

(1) You need to invert a matrix $\underline{\underline{\Phi}}$ for all values of t

(2) You need to multiply this matrix with a vector of functions and then integrate each element of that vector.
(lots of work)

(3) In the end you need to multiply all of that by another matrix of functions...

EXAMPLE Find $\underline{x_p}$ for $\underline{x}' = \begin{bmatrix}4 & 2\\ 3 & -1\end{bmatrix}\underline{x} - \begin{bmatrix}1\\ 2\end{bmatrix} t e^{-2t}$, given that

$$\underline{x_H} = C_1 \begin{bmatrix}1\\ -3\end{bmatrix} e^{-2t} + C_2 \begin{bmatrix}2\\ 1\end{bmatrix} e^{5t}$$

1. Set up $\underline{\underline{\Phi}}(t)$ and $(\underline{\underline{\Phi}}(t))^{-1}$

$$\underline{\underline{\Phi}}(t) = \begin{bmatrix} e^{-2t} & 2e^{5t}\\ -3e^{-2t} & e^{5t}\end{bmatrix} = e^{5t}\begin{bmatrix} e^{-7t} & 2\\ -3e^{-7t} & 1\end{bmatrix}$$

so

$$(\underline{\underline{\Phi}}(t))^{-1} = \frac{1}{e^{3t} + 6e^{3t}} \begin{bmatrix} e^{5t} & -2e^{5t}\\ -3e^{-2t} & e^{-2t}\end{bmatrix} = \frac{1}{7}\begin{bmatrix} e^{2t} & -2e^{2t}\\ -3e^{-5t} & e^{-5t}\end{bmatrix}$$

2. Calc $(\underline{\underline{\Phi}}(t))^{-1}\underline{f}(t)$

$$\frac{e^{-5t}}{7}\begin{bmatrix} e^{7t} & -2e^{7t}\\ -3 & 1\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix} t e^{-2t} = \frac{te^{-7t}}{7}\begin{bmatrix} -3e^{7t}\\ -1\end{bmatrix} = \begin{bmatrix} -\frac{3}{7}t\\ -\frac{t}{7}e^{-7t}\end{bmatrix}$$

3. calc $\int (\underline{\underline{\Phi}}(t))^{-1}\underline{f}(t)\,dt$

$$\int \underline{\underline{\Phi}}(t)\underline{f}(t)\,dt = \begin{bmatrix} -\frac{3}{14}t^2\\ \frac{te^{-7t}}{49} + \frac{e^{-7t}}{7^3}\end{bmatrix}$$

4. calc $\underline{\underline{\Phi}}(t)\int (\underline{\underline{\Phi}}(t))^{-1}\underline{f}(t)\,dt$

$$\underline{x_p} = e^{5t}\begin{bmatrix} e^{-7t} & 2\\ -3e^{-7t} & 1\end{bmatrix}\begin{bmatrix} -\frac{3}{14}t^2\\ \frac{te^{-7t}}{49} + \frac{e^{-7t}}{7^3}\end{bmatrix} = e^{5t}\begin{bmatrix} -\frac{3}{14}t^2e^{-7t} + \frac{2t}{49}e^{-7t} + \frac{2}{7^3}e^{-7t}\\ \frac{9}{14}t^2e^{-7t} + \frac{t}{49}e^{-7t} + \frac{e^{-7t}}{7^3}\end{bmatrix}$$

$$= \boxed{\left(\frac{1}{7^3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{7^2} \begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \frac{3}{14} \begin{bmatrix} -1 \\ 3 \end{bmatrix} t^2 \right) e^{-2t}}$$

Note: Big huge chance I made an error in my calculations here...

CHAPTER 7. LAPLACE TRANSFORM METHODS

SEC 7.1. LAPLACE TRANSFORMS & INVERSE LAPLACE TRANSFORMS

DEFINITION The Laplace transform is a linear operator that maps nice functions to functions in the following way

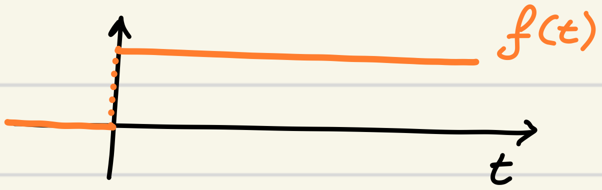
$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

Why care? Often in engineering one encounters DEs of the form

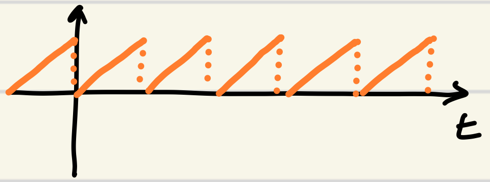
$$L(y) = f(t)$$

where $f(t)$ is discontinuous.

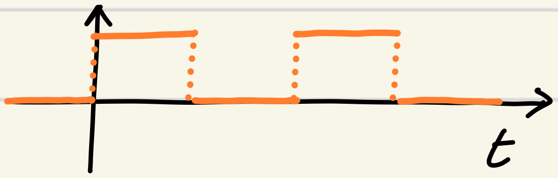
EXAMPLES (1) $f \leftrightarrow$ voltage turned on by flipping switch at $t = 0$



(2) When sending electric signals, often a sawtooth voltage



or a block wave voltage



is used.

In these cases the Laplace transform (LT) can be useful. The idea behind the LT method is the following:

Given an IVP of the following form:

$$\begin{cases} \sum_{i=0}^{n} a_i x^{(i)}(t) = f(t) \\ x(0)=b_0,\ x'(0)=b_1, \dots, x^{(n-1)}(0)=b_{n-1} \end{cases} \qquad \circledast$$

where the a_i are constant.

Then applying the LT to e.g. x' gives us

$$\mathcal{L}\{x'(t)\} = \int_0^\infty x'(t)e^{-st}dt \overset{\substack{\text{integration} \\ \text{by parts}}}{=} [x(t)e^{-st}]_0^\infty + s\int_0^\infty x(t)e^{-st}$$

$$\overset{\text{if } x \text{ doesn't grow too fast}}{=} -x(0) + s\,\mathcal{L}\{x(t)\}$$

$$= sX(s) - b_0 \qquad \text{with } X(s) := \mathcal{L}\{x(t)\}$$

So the LT converts a derivative of x to a polynomial in s. Higher derivatives are converted to higher degree polynomials in s. Applying the LT to both sides of ⊛ results in

$$X(s)\,\overset{\substack{n^{th}\text{ degree}\\ \text{pol in } s\\ \downarrow}}{Pol_1(s)} + pol_2(s) = \overset{\substack{F(s)\\ //}}{\mathcal{L}\{f(t)\}}$$

$$\Rightarrow X(s) = \frac{F(s) - pol_2(s)}{pol_1(s)}$$

$$\Rightarrow x(t) = \mathcal{L}^{-1}\left\{\frac{F(s) - pol_2(s)}{pol_1(s)}\right\}$$

Notes: (1) This is only true if the coeffs a_i are constants!

(2) We will often break some mathematical rules like:

- assume $x(t)$ grows moderately
- divide by 0
- use non existing real functions.

This is perfectly fine though: because of uniqueness & existence we know that a linear IVP with const coeffs only has 1 solution so we can use any method to find it, even if the method is shady.

EXAMPLES

① $$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{at}e^{-st}\,dt = \int_0^\infty e^{(a-s)t}\,dt$$

$$= \frac{1}{a-s}\left[e^{(a-s)t}\right]_0^\infty$$

$$= \begin{cases} \frac{1}{s-a} & s > a \\ \text{undefined} & a \leq s \end{cases}$$

In particular $\mathcal{L}\{\overset{\substack{\text{const}\\ \downarrow}}{c}\} = c\mathcal{L}\{1\} = c\mathcal{L}\{e^{0t}\} = \frac{c}{s}$ for $s>0$

② $\mathcal{L}\{\cosh(kt)\} = \mathcal{L}\left\{\frac{e^{kt}+e^{-kt}}{2}\right\}$

$$= \frac{1}{2}\left(\mathcal{L}\{e^{kt}\} + \mathcal{L}\{e^{-kt}\}\right)$$

$$= \frac{1}{2}\left(\frac{1}{s-k} + \frac{1}{s+k}\right) \qquad s>|k|$$

$$= \frac{s}{s^2-k^2}$$

(3) $\mathcal{L}\{\sinh(kt)\} = \mathcal{L}\left\{\frac{e^{kt}-e^{-kt}}{2}\right\} = \frac{1}{2}\left(\frac{1}{s-k} - \frac{1}{s+k}\right) \quad s>|k|$

$$= \frac{k}{s^2-k^2}$$

④ $\mathcal{L}\{\cos(kt)\} = \mathcal{L}\left\{\frac{e^{ikt}+e^{-ikt}}{2}\right\} = \mathcal{L}\{\cosh(ikt)\}$

$= \frac{s}{s^2+k^2}$ (no restrictions on s here)

⑤ $\mathcal{L}\{\sin(kt)\} = \mathcal{L}\left\{\frac{e^{ikt}-e^{-ikt}}{2i}\right\} = \frac{1}{i}\mathcal{L}\{\sinh(ikt)\}$

$= \frac{k}{s^2+k^2}$ (no restrictions on s here)

⑥ $\mathcal{L}\{t^{\overset{\substack{a>-1\\ \downarrow}}{a}}\} = \int_0^\infty t^a e^{-st}\,dt$

Let $u = \overset{s>0}{st} \Rightarrow du = s\,dt \Rightarrow dt = \frac{du}{s}$

$$= \int_0^\infty \left(\frac{u}{s}\right)^a e^{-u} \frac{du}{s} = \frac{1}{s^{a+1}} \int_0^\infty u^a e^{-u} du =: \frac{\Gamma(a+1)}{s^{a+1}} \quad s>0$$

where

$$\Gamma(a+1) := \int_0^\infty u^a e^{-u} du$$

is called the gamma function. It is a generalization of the factorial function since for $a>0$

$$\Gamma(a+1) = \int_0^\infty u^a e^{-u} du = -\int_0^\infty u^a d e^{-u} = -\left[u^a e^{-u}\right]_0^\infty + a\int_0^\infty u^{a-1} e^{-u} du$$

$$\overset{a>0}{=} a\Gamma(a)$$

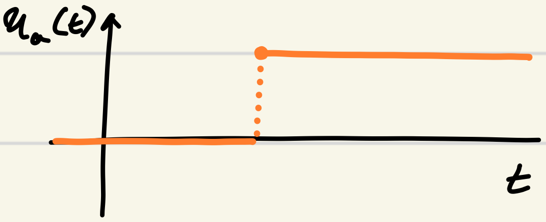
Moreover $\Gamma(1) = \int_0^\infty e^{-u} du = -\left[e^{-u}\right]_0^\infty = 1$

So for a a non negative integer

$$\Gamma(a+1) = a!$$

The LT can also be taken of a piecewise continuous function. Consider, e.g., the unit step function

$$u_a(t) = \begin{cases} 1 & t \geq a \\ 0 & t < a \end{cases}$$



then $$\mathcal{L}\{u_a(t)\} = \int_0^\infty u_a(t)\, e^{-st}\, dt$$

if $a>0$ then

since $u_a(t) = 0$ for $t<a$

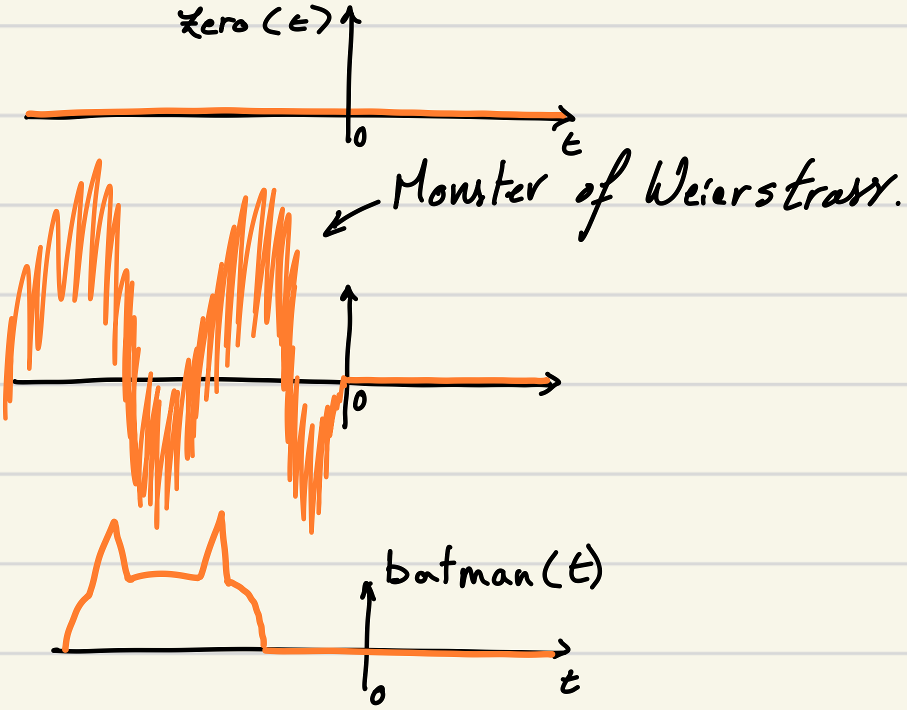
$$= \cancel{\int_0^a u_a(t)\, e^{-st}\, dt} + \int_a^\infty \underset{1}{\cancel{u_a}}(t)\, e^{-st}\, dt$$

$$= -\frac{1}{s}\left[e^{-st}\right]_a^\infty$$

$$= \frac{e^{-as}}{s} \qquad s>0$$

Note that the LT does not exist for all functions. For example $\int_0^\infty e^{t^2} e^{-st}\, dt$ diverges so $\mathcal{L}\{e^{t^2}\}$ does not exist.

The LT is only invertible for functions defined on the nonnegative real line. This is because it ignores values of functions for negative values of t. The following all have the same LT:



7.2. TRANSFORMS OF IVPs

The LT can be used to transform a linear ivp with constant coeffs in the unknown function $x(t)$ into an equation (thats trivial to solve) in the unknown $X(s)$. Indeed, given an IVP

$$\begin{cases} \sum_{i=0}^{n} a_i x^{(i)}(t) = f(t) \\ x(0) = b_0, x'(0) = b_1, \dots, x^{(n-1)}(0) = b_{n-1} \end{cases}$$

then, since $\mathcal{L}\{x'\} = s\mathcal{L}\{x\} - x(0)$ we have that

$$\begin{aligned} \mathcal{L}\{x''\} &= \mathcal{L}\{(x')'\} = s\mathcal{L}\{x'\} - x'(0) \\ &= s(s\mathcal{L}\{x\} - x(0)) - x'(0) \\ &= s^2\mathcal{L}\{x\} - s x(0) - x'(0) \\ \mathcal{L}\{x'''\} &= \mathcal{L}\{(x'')'\} = s\mathcal{L}\{x''\} - x''(0) \\ &= s(s^2\mathcal{L}\{x\} - s x(0) - x'(0)) - x''(0) \\ &= s^3\mathcal{L}\{x\} - s^2 x(0) - s x'(0) - x''(0) \end{aligned}$$

In general

$$\mathcal{L}\{x^{(n)}\} = s^n\mathcal{L}\{x\} - s^{n-1}x(0) - s^{n-2}x'(0) - \dots - s x^{(n-2)}(0) - x^{(n-1)}(0)$$

or, if we write $\mathcal{L}\{x\} = X(s)$, and $x(0) = b_0, \dots, x^{(n-1)}(0) = b_{n-1}$

$$\boxed{\mathcal{L}\{x^{(n)}\} = s^n X(s) - s^{n-1}b_0 - s^{n-2}b_1 - \dots - s b_{n-2} - b_{n-1}}$$

EXAMPLES (1) Solve

$$\begin{cases} x'' - x' - 6x = 0 \\ x(0) = 2, \ x'(0) = -1 \end{cases}$$

using the LT.

Solution:

$$-6\mathcal{L}\{x\} = -6X(s) \qquad \mathcal{L}\{0\} = 0$$

$$-\mathcal{L}\{x'\} = -sX(s) + 2$$

$$\mathcal{L}\{x''\} = s^2X(s) - 2s + 1$$

$$\Rightarrow \quad X(s)(s^2 - s - 6) - 2s + 3 = 0$$

$$\Rightarrow \quad X(s) = \frac{2s-3}{s^2-s-6}$$

$$\Rightarrow \quad x(t) = \mathcal{L}^{-1}\left\{\frac{2s-3}{s^2-s-6}\right\}$$

To proceed we will try to write $\frac{2s-3}{s^2-s-6}$ as a sum of fractions:

$$\frac{2s-3}{s^2-s-6} = \frac{2s-3}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$$

$$= \frac{A(s+2) + B(s-3)}{(s-3)(s+2)}$$

$$= \frac{s(A+B) + 2A - 3B}{(s-3)(s+2)}$$

$$\Rightarrow \begin{cases} A + B = 2 \\ 2A - 3B = -3 \end{cases} \Rightarrow \begin{cases} 5A = 3 \\ -5B = -7 \end{cases} \Rightarrow \begin{cases} A = \frac{3}{5} \\ B = \frac{7}{5} \end{cases}$$

$$\Rightarrow \quad \frac{2s-3}{s^2-s-6} = \frac{3}{5}\frac{1}{s-3} + \frac{7}{5}\frac{1}{s+2}$$

$$\Rightarrow \quad x(t) = \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{7}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= \boxed{\frac{3}{5}e^{3t} + \frac{7}{5}e^{-2t}}$$

(2) Solve $\begin{cases} x'' + 4x = \sin(3t) \\ x(0) = x'(0) = 0 \end{cases}$ using the LT

$$4\mathcal{L}\{x\} = 4X(s) \qquad \mathcal{L}\{\sin(3t)\} = \frac{3}{s^2+9}$$

$$\mathcal{L}\{x''\} = s^2X(s)$$

$$\Rightarrow \quad X(s)(s^2+4) = \frac{3}{s^2+9}$$

$$\Rightarrow \quad X(s) = \frac{3}{(s^2+4)(s^2+9)}$$

We will use partial fraction decomposition:

$$\frac{3}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

$$= \frac{(As+B)(s^2+9) + (Cs+D)(s^2+4)}{(s^2+4)(s^2+9)}$$

$$= \frac{s^3(A+C) + s^2(B+D) + s(9A+4C) + 9B+4D}{(s^2+4)(s^2+9)}$$

$$\Rightarrow \begin{cases} A+C=0 \\ B+D=0 \\ 9A+4C=0 \\ 9B+4D=3 \end{cases} \Rightarrow \begin{cases} A=C=0 \\ 5B=3 \\ -5D=3 \end{cases} \Rightarrow \quad A=C=0,\ B=\frac{3}{5},\ D=-\frac{3}{5}$$

so $$X(s) = \frac{3}{5}\frac{1}{s^2+4} - \frac{3}{5}\frac{1}{s^2+9} = \frac{3}{10}\frac{2}{s^2+2^2} - \frac{1}{5}\frac{3}{s^2+3^2}$$

$$\Rightarrow \boxed{X(t) = \frac{3}{10}\sin(2t) - \frac{1}{5}\sin(3t)}$$

ADDITIONAL TRANSFORM TECHNIQUES

Using the fact that $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$ we can easily calculate $\mathcal{L}\{te^{at}\}$ without integration.

Indeed, let $f(t) = te^{at}$, then $f'(t) = e^{at} + ate^{at}$

so $$\underset{\overset{\parallel}{\mathcal{L}\{e^{at}\} + aF(s)}}{\mathcal{L}\{f'(t)\}} = sF(s) - f(0) = sF(s)$$

$$\Rightarrow \frac{1}{s-a} + aF(s) = sF(s) \qquad s > a$$

$$\Rightarrow F(s)(s-a) = \frac{1}{s-a} \qquad s > a$$

$$\Rightarrow F(s) = \frac{1}{(s-a)^2} \qquad s > a$$

Likewise for $\mathcal{L}\{t\sin(at)\}$:

let $f(t) = t\sin(at)$ then $f'(t) = \sin(at) + at\cos(at)$

so $$\mathcal{L}\{f'(t)\} = sF(s)$$
$$= \mathcal{L}\{\sin(at)\} + a\mathcal{L}\{t\cos(at)\}$$

$$\Rightarrow sF(s) = \frac{a}{s^2+a^2} + a\mathcal{L}\{t\cos(at)\}$$

Let $g(t) = t\cos(at)$ then $g'(t) = \cos(at) - at\sin(at)$

so $$\mathcal{L}\{g'(t)\} = sG(s)$$
$$= \frac{s}{s^2+a^2} - aF(s)$$

$$\text{so}\begin{cases} sF(s) = \dfrac{a}{s^2+a^2} + aG(s) & ① \\ sG(s) = \dfrac{s}{s^2+a^2} - aF(s) & ② \end{cases}$$

$$\Rightarrow \begin{cases} s^2F(s) = \dfrac{as}{s^2+a^2} + \dfrac{as}{s^2+a^2} - a^2F(s) \\ s^2G(s) = \dfrac{s^2}{s^2+a^2} - \dfrac{a^2}{s^2+a^2} - a^2G(s) \end{cases}$$

$$\Rightarrow \begin{cases} F(s)(s^2+a^2) = \dfrac{2as}{s^2+a^2} \\ G(s)(s^2+a^2) = \dfrac{s^2-a^2}{s^2+a^2} \end{cases}$$

$$\Rightarrow \begin{cases} F(s) = \dfrac{2as}{(s^2+a^2)^2} = \mathcal{L}\{t\sin(at)\} \\ G(s) = \dfrac{s^2-a^2}{(s^2+a^2)^2} = \mathcal{L}\{t\cos(at)\} \end{cases}$$

just like $\mathcal{L}\{f'(t)\}$ has a nice form we also have that

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}$$

has a nice form:

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{F(s)}{s}$$

and also

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\,d\tau = \int_0^t \left(\mathcal{L}^{-1}\{F(s)\}\right)(\tau)\,d\tau$$

EXAMPLE Find the inverse LT of $\frac{1}{s^2(s-a)}$.

$$\frac{1}{s^2(s-a)} = \frac{\frac{1}{s(s-a)}}{s} \quad \text{so} \quad \mathcal{L}^{-1}\left\{\frac{1}{s^2(s-a)}\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{1}{s(s-a)}}{s}\right\}$$

$$= \int_0^t \left(\mathcal{L}^{-1}\left\{\frac{1}{s(s-a)}\right\}\right)(\tau)\,d\tau$$

But $\left(\mathcal{L}^{-1}\left\{\frac{1}{s(s-a)}\right\}\right)(\tau) = \int_0^\tau \left(\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}\right)(T)\,dT$

$$= \int_0^\tau e^{aT}\,dT = \frac{1}{a}\left[e^{aT}\right]_0^\tau = \frac{1}{a}\left(e^{a\tau}-1\right)$$

$$= \frac{1}{a}\int_0^t \left(e^{a\tau}-1\right)d\tau = \frac{1}{a^2}\left[e^{a\tau}\right]_0^t - \left[\frac{\tau}{a}\right]_0^t = \frac{e^{at}-1}{a^2} - \frac{t}{a}$$

$$= \boxed{\frac{e^{at}}{a^2} - \frac{t}{a} - \frac{1}{a^2}}$$

SEC 7.3 PARTIAL FRACTIONS

Often we need to take the inverse LT of something of the form

$$R(s) = \frac{P(s)}{Q(s)} \; \Big] \text{Polynomials}$$

with degree (P) < degree (Q)

To find $\mathcal{L}^{-1}\{R(s)\}$ we need the method of partial fractions.

This method works as follows:

1> Factorize Q in irreducible linear and quadratic factors as follows:

$$Q(s) = \underbrace{(s-a_1)^{m_1}(s-a_2)^{m_2}\cdots(s-a_k)^{m_k}}_{\substack{k \text{ linear irreducible} \\ \text{factors, each with} \\ \text{multiplicity } n_i}}\underbrace{((s-b_1)^2+c_1^2)^{n_1}\cdots((s-b_l)^2+c_l^2)^{n_l}}_{\substack{l \text{ quadratic irreducible} \\ \text{factors, each with} \\ \text{multiplicity } n_i}}$$

2> Next we write $\frac{P(s)}{Q(s)}$ as a sum of fractions as follows

- For each linear factor (of the form $(s-a)^m$) of Q, construct a sum of m partial fractions:

$$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_m}{(s-a)^m}$$

where $A_1, \ldots, A_m$ are unknown constants.

▶ For each quadratic factor (of the form $((s-b)^2+c^2)^n$) of Q, construct a sum of m partial fractions:

$$\frac{B_1 s + C_1}{(s-b)^2+c^2} + \frac{B_2 s + C_2}{((s-b)^2+c^2)^2} + \dots + \frac{B_n s + C_n}{((s-b)^2+c^2)^n}$$

with $B_1, \dots, B_n, C_1, \dots, C_n$ unknown constants.

$\underline{3}$ > Add all partial fractions together and demand that this sum equals $R(s)$ for all values of s. This will result in a linear system of equations in the various unknowns A_i, B_i, C_i whose solution determines the values of these constants.

After we convert $R(s) = \frac{P(s)}{Q(s)}$ to a sum of terms of the form

$$\underbrace{\frac{A}{(s-a)^k}}_{\text{CASE 1}} \quad \text{and} \quad \underbrace{\frac{Bs + C}{((s-b)^2+c^2)^k}}_{\text{CASE 2}}$$

we want to compute the inverse LT of those fractions. The following observation is useful for this purpose:

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{at} f(t)\, e^{-st} dt = \int_0^\infty f(t)\, e^{-(s-a)t} dt$$

$$= F(s-a)$$

So also $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\mathcal{L}^{-1}\{F(s)\}$

CASE 1

We know that $\mathcal{L}\{t^{k-1}\} = \dfrac{\Gamma(k)}{s^k}$

so

$$\mathcal{L}\{e^{at}t^{k-1}\} = \frac{(k-1)!}{(s-a)^k}$$

$$\Leftrightarrow \quad \frac{\mathcal{L}\{e^{at}t^{k-1}\}}{(k-1)!} = \frac{1}{(s-a)^k}$$

$$\Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{1}{(s-a)^k}\right\} = \frac{e^{at}t^{k-1}}{(k-1)!}$$

CASE 2

We can only work out case 2 when $k=1$ at this moment. In sec 7.4 we'll see how to do the general case.

If $k=1$ we want to find

$$\mathcal{L}^{-1}\left\{\frac{Bs+G}{(s-b)^2+c^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{B(s-b)+Bb+G}{(s-b)^2+c^2}\right\}$$

$$= B\mathcal{L}^{-1}\left\{\frac{(s-b)}{(s-b)^2+c^2}\right\} + \frac{Bb+G}{c}\mathcal{L}^{-1}\left\{\frac{c}{(s-b)^2+c^2}\right\}$$

$$= B\,e^{bt}\cos(ct) + \frac{Bb+G}{c}\,e^{bt}\sin(ct)$$

7.4. PRODUCTS, DERIVATIVES & INTEGRALS OF TRANSFORMS

Often $X(s)$ can be recognized as a product of 2 factors $F(s)\,G(s)$ whose inverse LT is known.

EXAMPLE

$$\begin{cases} x'' + x = \cos(t) \\ x(0) = x'(0) = 0 \end{cases}$$

then

$$\mathcal{L}\{x''\} = s^2 X(s) \qquad \mathcal{L}\{\cos(t)\} = \frac{s}{s^2+1}$$

$$\mathcal{L}\{x\} = X(s)$$

$$X(s)(1+s^2) = \frac{s}{s^2+1}$$

$$\Rightarrow \quad X(s) = \frac{s}{s^2+1}\,\frac{1}{s^2+1} = \mathcal{L}\{\cos(t)\}\,\mathcal{L}\{\sin(t)\}$$

$$\neq \mathcal{L}\{\cos(t)\sin(t)\} = \frac{1}{s^2+2^2}$$

There is, however, a product, $*$, such that

$$\mathcal{L}\{f\}\,\mathcal{L}\{g\} = \mathcal{L}\{f * g\}$$

So in particular, if

$$X(s) = \mathcal{L}\{f\}\,\mathcal{L}\{g\}$$

$$\Rightarrow \quad x(t) = \mathcal{L}^{-1}\{\mathcal{L}\{f\}\,\mathcal{L}\{g\}\} = f * g$$

DEFINITION $*$ is called the convolution product and is defined (for $t \geqslant 0$) as

$$(f*g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

EXAMPLE (continued)

$$X(s) = \mathcal{L}\{\cos(t)\}\, \mathcal{L}\{\sin(t)\}$$

$$\Rightarrow x(t) = (\cos * \sin)(t)$$

$$= \int_0^t \cos(\tau)\sin(t-\tau)\, d\tau$$

Since $\sin(A+B) = \sin(A)\cos(B) + \sin(B)\cos(A)$

$\sin(A-B) = \sin(A)\cos(B) - \sin(B)\cos(A)$

$$\frac{\sin(A+B)+\sin(A-B)}{2} = \sin(A)\cos(B)$$

So with $A = t-\tau$, $B = \tau$: $\cos(\tau)\sin(t-\tau) = \frac{1}{2}\left(\sin(t)+\sin(t-2\tau)\right)$

$$= \frac{1}{2}\int_0^t \sin(t)\, d\tau - \frac{1}{2}\int_0^t \sin(2\tau - t)\, d\tau$$

$$= \frac{t}{2}\sin(t) + \frac{1}{4}\Big[\cos(2\tau - t)\Big]_0^t$$

$$= \frac{t}{2}\sin(t) + \frac{1}{4}\left(\cos(t) - \cos(-t)\right)$$

$$= \frac{t\sin(t)}{2}$$

Notes

(1) $*$ is often defined as

$$(f*g)(t) = \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)\,d\tau$$

but since we only care about $t>0$ we set $f(t)=0$ and $g(t)=0$ for $t<0$ and we get

$$(f*g)(t) = \int_0^t \underset{\substack{\downarrow \\ 0 \text{ for } \tau<0}}{f(\tau)}\, \overset{\substack{0 \text{ for } t-\tau<0,\text{ i.e. } \tau>t \\ \downarrow}}{g}(t-\tau)\,d\tau$$

(2) Convolution is commutative

(3) Sometimes the notation $f(t)*g(t)$ is used. This can lead to all sorts of trouble.

For example: is

$$f(2t)*g(2t) \overset{?}{=} \int_0^{2t} f(\tau)g(t-\tau)\,d\tau$$

or

$$f(2t)*g(2t) \overset{?}{=} \int_0^{t} f(2\tau)g(2t-2\tau)\,d\tau$$

If confusion arises you can use the notation

$$(f(s)*g(s))(t)$$

or more sloppy:

$$(f(t)*g(t))(t)$$

so that

$$(f(2t)*g(2t))(t) = \int_0^t f(2\tau)g(2t-2\tau)\,d\tau$$

and $$(f(t)*g(t))(2t) = \int_0^{2t} f(\tau)\, g(t-\tau)\, d\tau$$

DIFFERENTIATION & INTEGRATION OF TRANSFORMS

Remember

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$
$$\mathcal{L}\{\int_0^t f(\tau)d\tau\} = \frac{F(s)}{s}$$

Are there similar formulas for

(A) $\mathcal{L}^{-1}\{F'(s)\}$

(B) $\mathcal{L}^{-1}\{\int_?^? F(\sigma)d\sigma\}$

(A) $$F'(s) = \frac{d}{ds}\int_0^\infty f(t)\, e^{-st} dt$$
$$= \int_0^\infty f(t) \frac{\partial}{\partial s}(e^{-st}) dt$$
$$= \int_0^\infty f(t)(-t)\, e^{-st}\, dt$$
$$= \mathcal{L}\{-tf(t)\}$$

So

$$\mathcal{L}\{-tf(t)\} = \frac{d}{ds}\left(\mathcal{L}\{f(t)\}\right)$$ useful for calc LT

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}$$ useful for calc ILT

EXAMPLE

(1) $\mathcal{L}^{-1}\left\{\tan^{-1}\left(\frac{1}{s}\right)\right\} = -\frac{1}{t}\,\mathcal{L}^{-1}\left\{\frac{1}{\left(\frac{1}{s}\right)^2+1}\left(\frac{-1}{s^2}\right)\right\}$

$$= \frac{1}{t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \frac{1}{t}\sin(t)$$

(B) If $\mathcal{L}\{-t f(t)\} = F'(s)$, then what does $\mathcal{L}\left\{\frac{f(t)}{t}\right\}$ give?

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_0^\infty \frac{f(t)\,e^{-st}}{t}\,dt$$

$$= \int_0^\infty f(t)\left(\int_s^\infty e^{-\sigma t}\,d\sigma\right)dt$$

$$= \int_s^\infty \left(\int_0^\infty f(t)\,e^{-\sigma t}\,dt\right)d\sigma$$

$$= \int_s^\infty F(\sigma)\,d\sigma$$

So

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \left(\mathcal{L}\{f(t)\}\right)(\sigma)\,d\sigma$$ useful for calc LT

$\mathcal{L}^{-1}\{$

7.5. PERIODIC & PIECEWISE CONTINUOUS INPUT FUNCTIONS

Previously we saw that

$$\mathcal{L}\{e^{at}f(t)\} = \underset{\text{F but translated to the right by a}}{\underset{\shortparallel}{F(s-a)}}$$

There also exists a similar law for the inverse LT:

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = \mathcal{L}^{-1}\{e^{-as}\int_0^\infty e^{-s\tau}f(\tau)\,d\tau\}$$

$$= \mathcal{L}^{-1}\{\int_0^\infty e^{-s(\tau+a)}f(\tau)\,d\tau\}$$

set $a+\tau = t \Rightarrow d\tau = dt$

$$= \mathcal{L}^{-1}\{\int_a^\infty f(t-a)e^{-st}\,dt\}$$

$$= \mathcal{L}^{-1}\{\int_0^\infty \underset{u_a(t)}{\underset{\shortparallel}{u(t-a)}}f(t-a)\,e^{-st}\,dt\}$$

$$= \mathcal{L}^{-1}\{\mathcal{L}\{u(t-a)f(t-a)\}\}$$

$$= u(t-a)f(t-a)$$

$$= u(t-a)\,\mathcal{L}^{-1}\{F(s)\}(t-a)$$

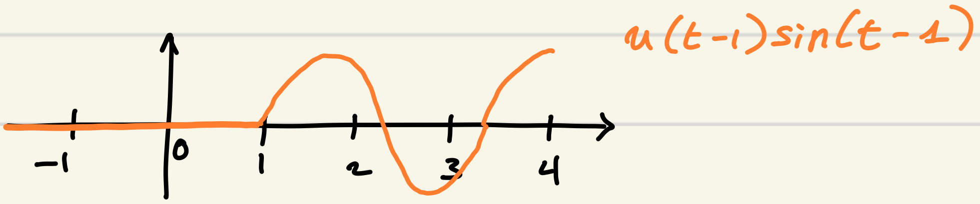
We also have the reverse rule:

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}\,\mathcal{L}\{f(t)\}$$

EXAMPLES

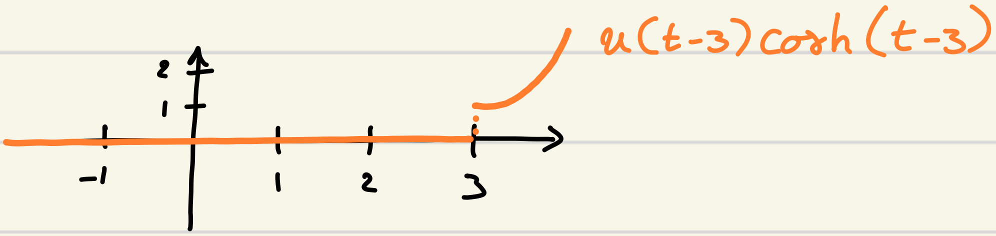
(1) $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+1}\right\} = u(t-1)\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t-1)$

$$= u(t-1)\sin(t-1)$$



(2) Let

$$g(t) = \begin{cases} 0 & t<3 \\ \cosh(t-3) & t\geq 3 \end{cases}$$

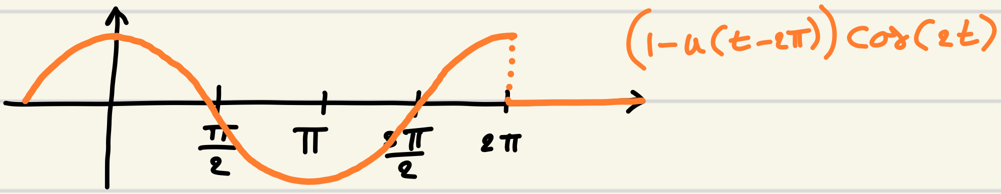


Then $\mathcal{L}\{g(t)\} = \mathcal{L}\{u(t-3)\cosh(t-3)\}$

$$= e^{-3s}\mathcal{L}\{\cosh(t)\}$$

$$= e^{-3s}\frac{s}{s^2-1}$$

(3) Find $\mathcal{L}\{f\}$ where $f(t) = \begin{cases} \cos(2t) & 0<t<2\pi \\ 0 & 2\pi<t \end{cases}$



Note that for $t>0$ $f(t) = (1-u(t-2\pi))\cos(2t)$

so $\mathcal{L}\{f(t)\} = \mathcal{L}\{\cos(2t) - u(t-2\pi)\cos(2t)\}$

$$= \frac{s}{s^2+4} - \mathcal{L}\{u(t-2\pi)\cos(2(t-2\pi)+\cancel{4\pi})\}$$

cos doesn't care about $+4\pi$

$$= \frac{s}{s^2+2^2} - e^{-2\pi s}\frac{s}{s^2+2^2} = \frac{s}{s^2+2^2}\left(1-e^{-2\pi s}\right)$$

LT's for periodic functions

For periodic functions, i.e. functions for which $f(t)=f(t+p)$ for some p the LT can be simplified as follows

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \sum_{n=0}^{\infty} \int_{np}^{(n+1)p} f(t)\, e^{-st}\, dt$$

Now let $\tau = t-np$, so $d\tau = dt$ and if $\begin{cases} t=(n+1)p \Rightarrow & \tau = p \\ t = np \Rightarrow & \tau = 0 \end{cases}$

$$= \sum_{n=0}^{\infty} \int_0^p f(\tau+np)\, e^{-s\tau - snp}\, d\tau$$

$$= \sum_{n=0}^{\infty} e^{-snp} \underbrace{\int_0^p f(\tau)\, e^{-s\tau}\, d\tau}$$

$$= \sum_{n=0}^{\infty} (e^{-sp})^n \int_0^p f(\tau)\, e^{-s\tau}\, d\tau$$

Let $\alpha = e^{-sP}$, then $\sum_{n=0}^{\infty}(e^{-sP})^n = \sum_{n=0}^{\infty}(\alpha)^n = 1 + \alpha + \alpha^2 + \dots$

Let

$$S(m) = \sum_{n=0}^{m} \alpha^n = 1 + \alpha + \dots + \alpha^m$$

then

$$\alpha S(m) = \alpha \sum_{n=0}^{m} \alpha^n = \alpha + \alpha^2 + \dots + \alpha^{m+1}$$

$$= 1 + \alpha + \dots + \alpha^m + \alpha^{m+1} - 1$$

$$= S(m) + \alpha^{m+1} - 1$$

So $(\alpha-1)S(m) = \alpha^{m+1}-1$

if $sp>0$, i.e. $0<\alpha<1$

$$S(m) = \frac{\alpha^{m+1}-1}{\alpha-1}$$

and thus

$$\sum_{n=0}^{\infty}\alpha^n = \lim_{m\to\infty} S(m) = \lim_{m\to\infty}\frac{\alpha^{m+1}-1}{\alpha-1} = \frac{1}{1-\alpha}$$

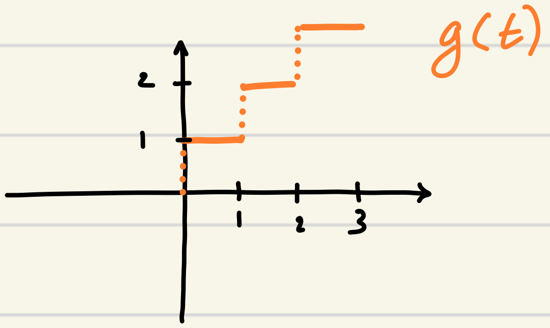
$$\Rightarrow \sum_{n=0}^{\infty} e^{-snp} = \frac{1}{1-e^{-sp}}$$

$\mathbb{L}$

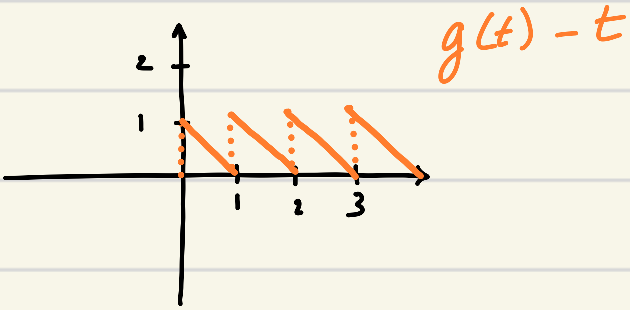
$$\Rightarrow \quad \boxed{\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sp}}\int_0^p f(t)\,e^{-st}\,dt}$$

EXAMPLE

Let $g(t) = 1 + \lfloor t \rfloor$ be the unit staircase function:



Let's calculate $\mathcal{L}\{1+\lfloor t\rfloor\}$. $g(t)$ is not periodic but $g(t)-t$ is:



so $\mathcal{L}\{g(t)\} = \mathcal{L}\{g(t)-t+t\} = \mathcal{L}\{g(t)-t\} + \frac{1}{s^2}$

and $\mathcal{L}\{g(t)-t\} = \frac{1}{1-e^{-s}} \int_0^1 (g(t)-t)\, e^{-st} dt$

$$= \frac{1}{1-e^{-s}} \int_0^1 (1-t) e^{-st} dt$$

$$= \frac{1}{1-e^{-s}} \frac{1}{(-s)} \int_0^1 (1-t)\, d(e^{-st})$$

$$= \frac{1}{1-e^{-s}} \left(\frac{1}{-s}\right) \left(\left[(1-t) e^{-st}\right]_0^1 + \int_0^1 e^{-st} dt \right)$$

$$= \frac{1}{1-e^{-s}} \frac{1}{s} \left(1 + \frac{1}{s}\left(e^{-s} - 1\right)\right)$$

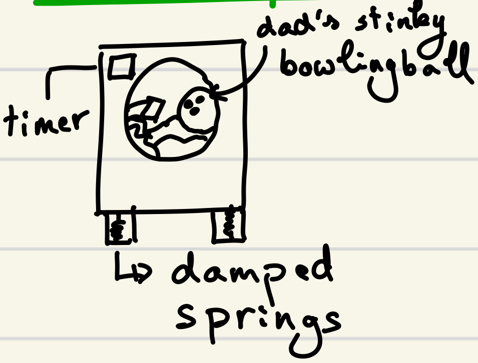
$$= \frac{1}{s(1-e^{-s})} - \frac{1}{s^2}$$

All together $\mathcal{L}\{g(t)\} = \frac{1}{s(1-e^{-s})} - \frac{1}{s^2} + \frac{1}{s^2} = \frac{1}{s(1-e^{-s})}$

7.6. THE DELTA FUNCTION

We've already seen how to describe systems where an external influence abrubtly takes effect at some time t using the step function.

EXAMPLE



When you set a delay timer on your washing machine, say for 6 a.m., then the vertical position of the system can be described by

$$mx'' + cx' + kx = \underset{\substack{\uparrow \\ \text{gravity}}}{mg} + u(t-6\text{a.m.})F_0\cos(\omega t)$$

But what if we want to describe a system that receives an instantaneous impulse at $t=a$? Consider, e.g., a golf ball thats hit by a bat in such a way that the momentum of the ball changes from 0 to 1. This means that the curve of the force, as a function of time, applied to the ball has surface area 1. At the same time we want the duration of the force to be 0s: should be instantaneous. With other words, we want $F(t)$ to have the following properties

(1) $$\int_{-\infty}^{+\infty} F(t)\,dt = 1$$

(2) $$F(t) = \begin{cases} 0 & t \neq a \\ ! & t = a \end{cases}$$

There is no real function with these properties since any function satisfying (2) must integrate to 0. Still, it would be very useful to have such a function so we're going to define it anyway.

DEFINITION The delta function is a symbol δ that satisfies

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{“}\infty\text{”} & t = 0 \end{cases}$$

and $\int_{-\infty}^{+\infty} \delta(t)\,dt = 1$. Its translation $\delta(t-a)$ is denoted by $\delta_a(t)$.

Rules for working with the δ function

(1) Do not talk about: $\frac{d}{dt}(\delta_a(t))$, $\overset{\text{not identity}}{\overset{\downarrow}{g}}(\delta_a(t))$, $\delta_a(\overset{\text{not translation}}{\overset{\downarrow}{g}}(t))$,

$\int_a^b \delta_a(t)(\ldots)\,dt$

↑ integration whose boundary hits the spike of $\delta_a(t)$.

(2) Let $t_0 < a < t_1$ then $\int_{t_0}^{t_1} \delta_a(t) f(t)\,dt = f(a)$

(3) $\int_{-\infty}^{t} \delta_a(\tau)\,d\tau = u_a(t)$ $\qquad \frac{d}{dt}(u_a(t)) = \delta_a(t)$

(4) For $a>0$, $\mathcal{L}\{\delta_a(t)\} = \int_0^{+\infty} \delta_a(t)e^{-st}\,dt = e^{-as}$

(5) For $a>0$, $(\delta_a * f)(t) = \int_0^t \delta_a(\tau) f(t-\tau)\,d\tau$

$$= \begin{cases} 0 & t<a \\ f(t-a) & t \geq a \end{cases}$$

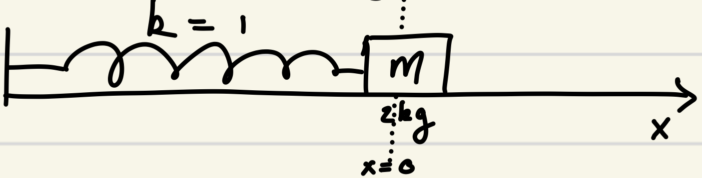
$$= u(t-a)f(t-a)$$

In particular: if $a=0$, $(\delta_a * f)(t) = u(t)f(t) = f(t)$ for $t>0$

so $\delta(t)$ is a unit for the convolution product.

EXAMPLES Applications of the δ function.

(1) A mass of 2kg is attached to a spring at rest with $k=1$ as follows



At $t=0$, $x=0$ and $x'=0$.

At $t=3$ an instantaneous force is applied that changes its momentum from 0 kg $\frac{m}{s}$ to +4 kg $\frac{m}{s}$. Find $x(t)$.

Newton's second law gives $m x'' = -kx + 4\delta(t-3)$

so the IVP is

$$\begin{cases} 2x'' + x = 4\delta(t-3) \\ x'(0) = x(0) = 0 \end{cases}$$

Applying $\mathcal{L}$ to the ivp:

$$\mathcal{L}\{2x''\} = 2s^2 X(s) \qquad \mathcal{L}\{4\delta(t-3)\} = 4e^{-3s}$$

$$\mathcal{L}\{x\} = X(s)$$

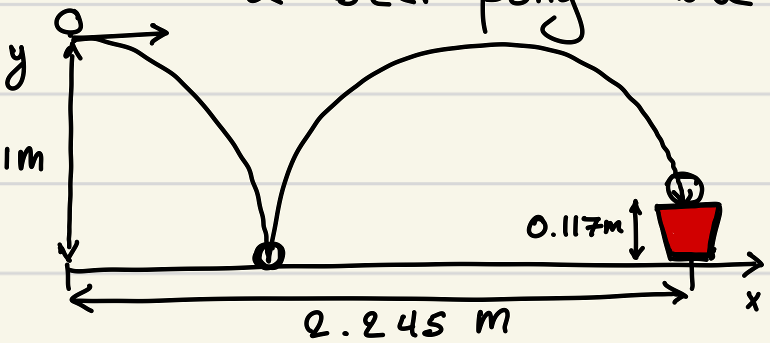
so $$X(s)(2s^2+1) = 4e^{-3s}$$

$\Rightarrow$ $$X(s) = 2\frac{e^{-3s}}{s^2+(\frac{1}{2})}$$

$\Rightarrow$ $$x(t) = 2\sin\left(\frac{1}{\sqrt{2}}(t-3)\right)$$

(2) The noble game of beer pong.

Given a beer pong table with the following setup.



If we thow a ping pong ball at hight $h=1$ with horizontal starting velocity V_x such that it lands exactly in the center of the cup. What function $(x(t), y(t))$ describes the trajectory of the ping pong ball between throw and landing in the cup?

Solution The functions $x(t)$ and $y(t)$ can be determined independently.

For $x(t)$: there is no horizontal force so $X(t) = V_x \cdot t$

Vertically: $y(0) = 1\,m$, $y'(0) = 0\,\frac{m}{s}$ and

$$my'' = -mg + F_{hit}\,\delta(t - t_{hit})$$

If we assume perfect elastic collision then the vertical momentum of the ball reverses upon hitting the table so $F_{hit} = \lim_{t \overset{<}{\to} t_{hit}} 2m|V_y(t)|$

What is $V_y(t)$ just before the hit? From conservation of energy:

$$\underset{\substack{\uparrow \\ \text{poten}}}{U_1} + \underset{\substack{\uparrow \\ \text{kinen}}}{T_1} = U_2 + T_2 \Rightarrow mg \cdot h_1 + \cancel{\tfrac{1}{2}mv_x^2} = \cancel{mg}\underset{\substack{\shortparallel \\ 0}}{h_2} + \cancel{\tfrac{1}{2}m}V_x^2 + \tfrac{1}{2}mv_y^2$$

so $mg = \frac{1}{2}mv_y^2 \Rightarrow v_y = -\sqrt{2g}$

so $F_{hit} = 2m\sqrt{2g}$

Next we want to know what t_{hit} is. Since the ball is free-falling before it hits the table: $y(t) = y_0 - \frac{1}{2}gt^2$

so, with $y(t_{hit}) = 0$, $y_0 = 1$, we get $\frac{1}{2}g t_{hit}^2 = 1 \Rightarrow t_{hit} = \sqrt{\frac{2}{g}}$

All together: $my'' = -gm + 2m\sqrt{2g}\,\delta(t - \sqrt{\tfrac{2}{g}})$

$$\Rightarrow \begin{cases} y'' = -g + 2m\sqrt{2g}\,\delta(t - \sqrt{\tfrac{2}{g}}) \\ y(0) = 1,\ y'(0) = 0 \end{cases}$$

$$\mathcal{L}\{y''\} = s^2Y(s) - y(0)s - \cancel{y'(0)} \qquad \mathcal{L}\{-g\} = \frac{-g}{s}$$

$$= s^2Y(s) - s \qquad \mathcal{L}\{2m\sqrt{2g}\,\delta(t - \sqrt{\tfrac{2}{g}})\} = 2m\sqrt{2g}\,e^{-\sqrt{\frac{2}{g}}s}$$

$$Y(s)s^2 - s = -\frac{g}{s} + 2m\sqrt{2g}\,e^{-\sqrt{\frac{2}{g}}s}$$

$$\Rightarrow Y(s) = -\frac{g}{s^3} + 2m\sqrt{2g}\,\frac{e^{-\sqrt{\frac{2}{g}}s}}{s^2} + \frac{1}{s}$$

$$\Rightarrow \qquad y(t) = -\frac{gt^2}{2} + 2m\sqrt{2g}\left(t - \sqrt{\tfrac{2}{g}}\right)u\left(t - \sqrt{\tfrac{2}{g}}\right) + 1$$

$$\text{so} \quad \big(x(t), y(t)\big) = \left(v_x t, -\frac{9.81}{2}t^2 + 8.86\,u(t-0.45)(t-0.45) + 1\right)$$

Homework: figure out v_x for a perfect throw.