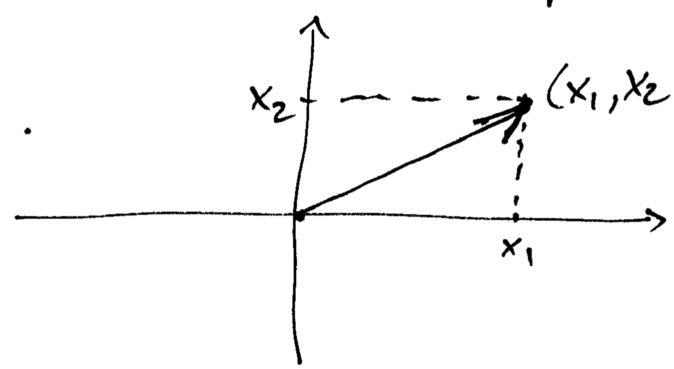
4.1–4.2. The vector space $\mathbb{R}^n$.

I – Vector calculus rules.

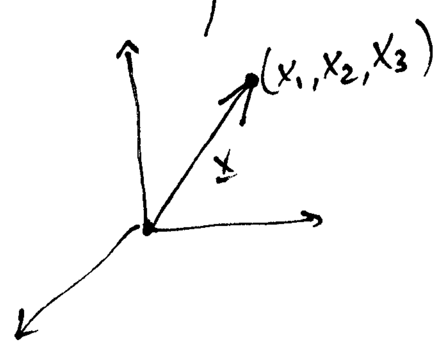
$\mathbb{R}^2$ = set of vectors w/ 2 real components

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$



$\mathbb{R}^3$ = ——————— 3 components.

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



$\mathbb{R}^n$ = ——————— n components

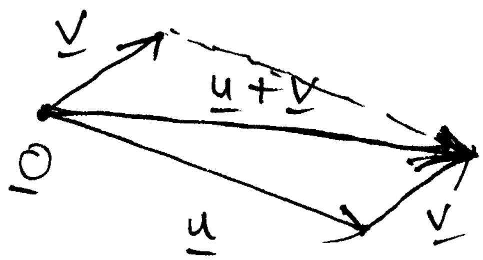
$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

<u>addition</u>: $\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}.$

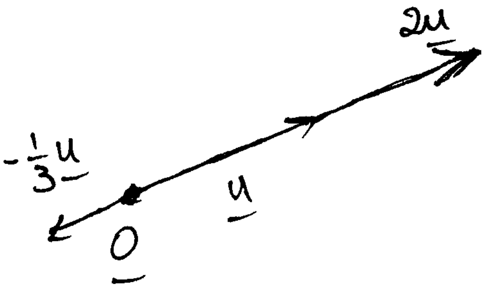
Scalar multiplication:

$$c\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} cv_1 \\ \vdots \\ cv_n \end{bmatrix}.$$

scalar
||
real number.



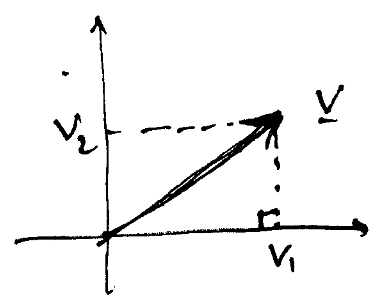
"parallelogram rule".



$$\underline{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

- **Length** (norm) of $\underline{v}$:

$$|\underline{v}| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$



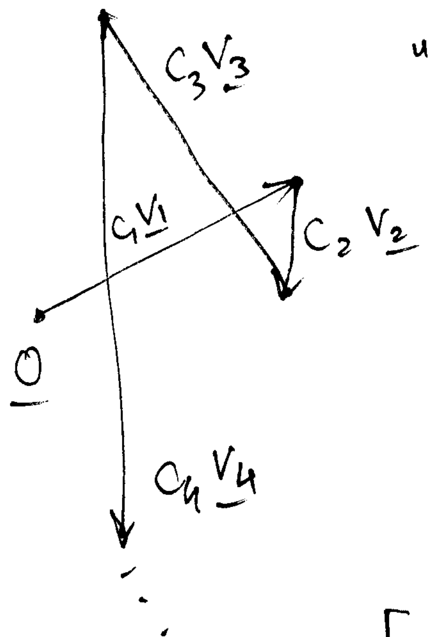
ex. $\underline{v} = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$, $|\underline{v}| = \sqrt{4+1+16}$
$= \sqrt{21}$.

II - Linear combinations

Let $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p$ be vectors in $\mathbb{R}^n$.
$c_1, c_2, \dots, c_p$ scalars.

Linear combination: $c_1 \underline{v}_1 + c_2 \underline{v}_2 + \cdots + c_p \underline{v}_p$.

"weights".



Ex. Is $\underline{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ a linear combination of $\underline{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\underline{v}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$?

Yes: $\underline{v} = \underline{v}_1 + 2\underline{v}_2$. $(c_1 = 1,\ c_2 = 2)$.

Def. $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p$ vectors in $\mathbb{R}^n$ are:

* <u>linearly dependent</u> if the equation

$$(\ast) \quad \boxed{c_1 \underline{v}_1 + c_2 \underline{v}_2 + \cdots + c_p \underline{v}_p = \underline{0}}$$

(in the variables $c_1, c_2, \dots, c_p$).

has a <u>non-trivial</u> solution.

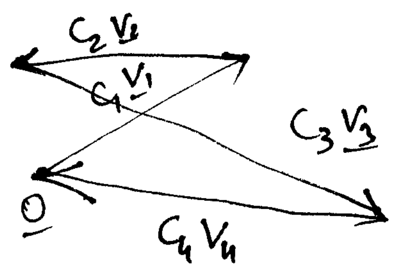
↳ $c_1, c_2, \dots, c_p$ are <u>not all</u> zero.

* <u>linearly independent</u> if the equation $(\ast)$ has only the trivial solution

$$c_1 = c_2 = \cdots = c_p = 0.$$

---

non-trivial lin. comb.:



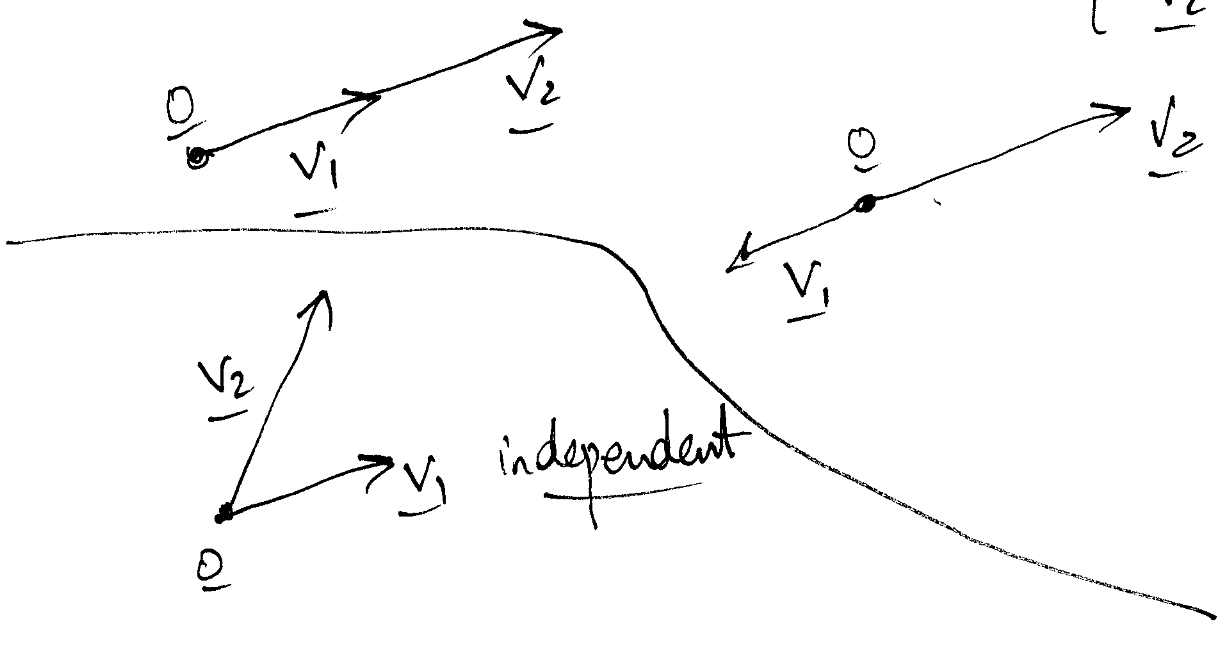
Rmk. Linearly dependent means that one of the vectors is a linear combination of the others:

ex. if $c_1 \neq 0$, then

$$\underline{v}_1 = -\frac{c_2}{c_1}\underline{v}_2 - \frac{c_3}{c_1}\underline{v}_3 - \dots - \frac{c_p}{c_1}\underline{v}_p .$$

Particular case: 2 vectors.

$\underline{v}_1$ and $\underline{v}_2$ are linearly dependent *iff* they are colinear:
$$\begin{cases} \underline{v}_1 = k_1 \underline{v}_2 \\ \text{or} \\ \underline{v}_2 = k_2 \underline{v}_1 . \end{cases}$$



<u>Example</u>. $\underline{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \underline{v}_3 = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

Are they linearly independent?

(*) becomes: $c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & 1 & 4 \\ 2 & 1 & -3 \end{bmatrix}}_{\underline{\underline{A}}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{\underline{x}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Row reduce $\underline{\underline{A}}$:

$$\underline{\underline{A}} \sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & -1 & -11 \end{bmatrix} \sim \begin{array}{ccc} \left[\begin{array}{ccc} \boxed{1} & 0 & -7 \\ 0 & \boxed{1} & 11 \end{array}\right] \\ \begin{array}{ccc} c_1 & c_2 & c_3\text{, free.} \end{array} \end{array}$$

$$\begin{cases} c_1 = 7c_3 \\ c_2 = -11c_3 \\ c_3 \text{ free} \end{cases}$$

$$\boxed{\underline{x} = \begin{bmatrix} 7c_3 \\ -11c_3 \\ c_3 \end{bmatrix} = c_3 \begin{bmatrix} 7 \\ -11 \\ 1 \end{bmatrix}} \quad c_3 \text{ free.}$$

$\underline{v}_1, \underline{v}_2, \underline{v}_3$ are linearly dependent.

Choose $c_3 = 1$: $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -11 \\ 1 \end{bmatrix}$.

hence $\boxed{7\underline{v}_1 - 11\underline{v}_2 + \underline{v}_3 = \underline{0}}$.

Example. $\underline{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\underline{v}_3 = \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix}$.

Are they linearly independent?

Can do the same : solve $\underline{\underline{A}} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

with $\underline{\underline{A}} = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 1 & -3 \\ 0 & 1 & -1 \end{bmatrix}$.

$\underline{\underline{A}}$ is square. Is it invertible?

Compute $\det \underline{\underline{A}}$.

$$\det \underline{A} = \begin{vmatrix} 1 & 1 & 4 \\ 0 & -1 & -11 \\ 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 4 \\ 0 & -1 & -11 \\ 0 & 0 & -12 \end{vmatrix} = 1 \cdot (-1)(-12) = 12 \neq 0.$$

Deduce that $\underline{A}$ invertible,

hence $\underline{A}\,\underline{x} = \underline{0}$ has only the trivial solution

$\underline{x} = \underline{0}$.

$\Longrightarrow$ $\underline{v}_1, \underline{v}_2, \underline{v}_3$ linearly independent.

Rmk. If $\underline{A}$ is a square matrix,

its columns are lin.-independent

iff $\underline{A}$ is invertible

iff $\det \underline{A} \neq 0$.

$\underline{v}_1, \underline{v}_2, \underline{v}_3$ above are an example of a basis for $\mathbb{R}^3$:

* they are linearly independent;
* for any $\underline{w}$ in $\mathbb{R}^3$, there exist c_1, c_2, c_3 such that $\underline{w} = c_1 \underline{v}_1 + c_2 \underline{v}_2 + c_3 \underline{v}_3$

Other basis for $\mathbb{R}^3$?

standard basis: $\underline{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\underline{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\underline{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$\left(\underline{\underline{I}}_3 = \left[\underline{i} \,|\, \underline{j} \,|\, \underline{k} \right] \text{ is invertible} \right)$.

III - Subspaces

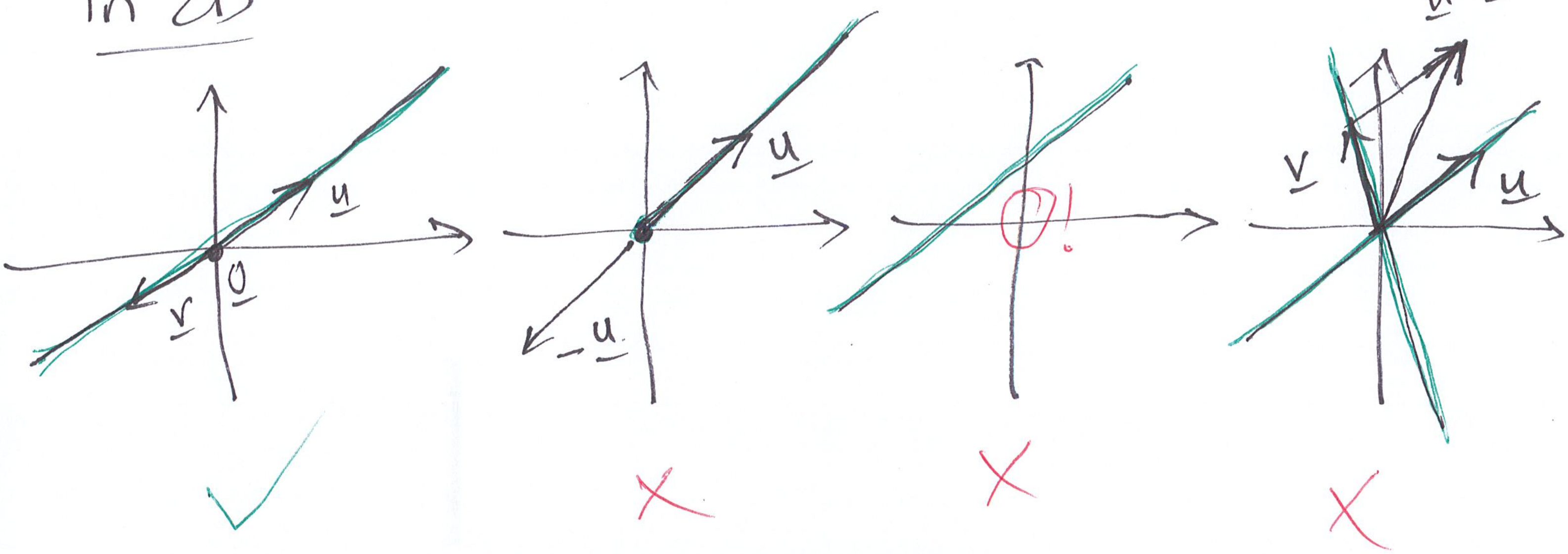
Def. a subset V of $\mathbb{R}^n$ is a <u>subspace</u> if

* it's non-empty (contains the zero vector)
* if $\underline{u}$ and $\underline{v}$ are in V, then $\underline{u} + \underline{v}$ is in V (V is closed / stable under addition)
* if $\underline{u}$ is in V and c is a scalar, then $c\underline{u}$ is in V. (V is stable under scalar mult.).

Ex. $\{\underline{0}\}$ and $\mathbb{R}^n$ "trivial" subspaces.

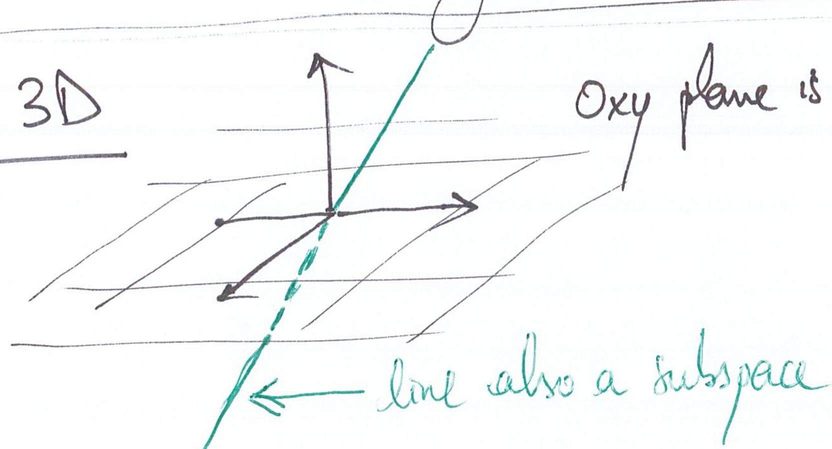
the other ones are called proper subspaces.

in 2D



proper subspace of $\mathbb{R}^2$ are lines going thru $\underline{0}$.

in 3D



Oxy plane is a proper subspace

Rmk. The solution set of $\underline{A}\underline{x} = \underline{0}$, for $\underline{A}_{m\times n}$, is a subspace of $\mathbb{R}^n$.