CHAPTER 3

LINEAR SYSTEMS & MATRICES

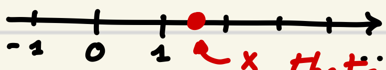
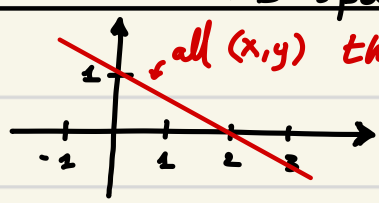
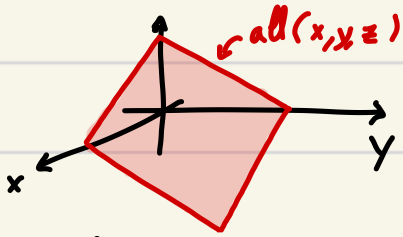
SEC 3.1. INTRO TO LINEAR SYSTEMS

DEFINITION A linear equation in the variables $x_1, x_2, \dots, x_n$ is of the form

KNOWN CONSTANTS

$$\underbrace{a_1}\, x_1 + \underbrace{a_2}\, x_2 + \dots + \underbrace{a_n}\, x_n = \underbrace{b}$$

EXAMPLES

#VARS	VAR NAMES	EQUATION	A GEOMETRICAL INTERPRETATION OF SOLN
1	x	$a_1x = b$	 Point in 1D space
2	x, y	$a_1x + a_2y = b$	 line in 2D space
3	x, y, z	$a_1x + a_2y + a_3z = b$	 Plane in 2D space

n	$x_1, \dots, x_n$	$a_1x_1 + \dots + a_nx_n = b$	$(n-1)$D plane in nD space

DEFINITION A system of linear equations in n variables is a finite collection of linear equations. Their solutions are n-tuples $(x_1, \dots, x_n)$ that satisfy all equations simultaneously.

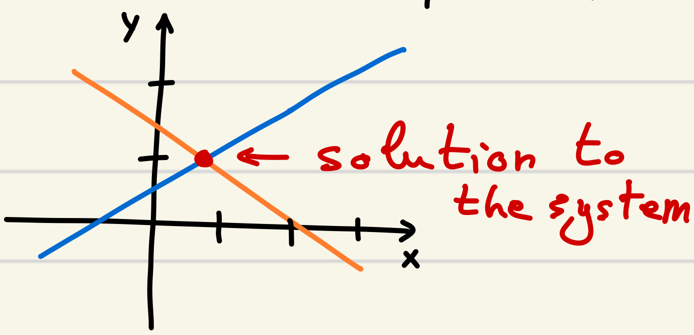
EXAMPLES

(1) A system of 2 lin eqns in 2 variables, say x, y, looks like

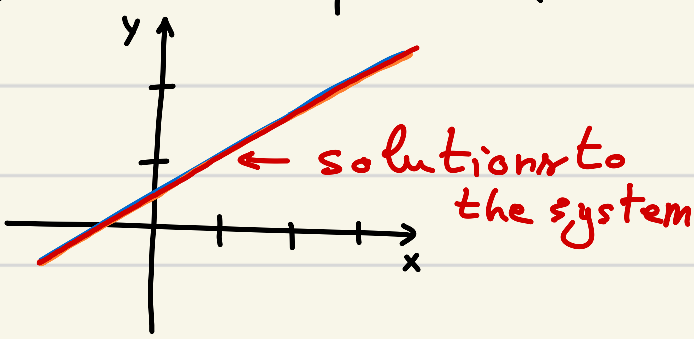
$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases}$$

The solutions to each of these form a straight line in the xy plane. There are 3 different scenarios for solutions to this system:

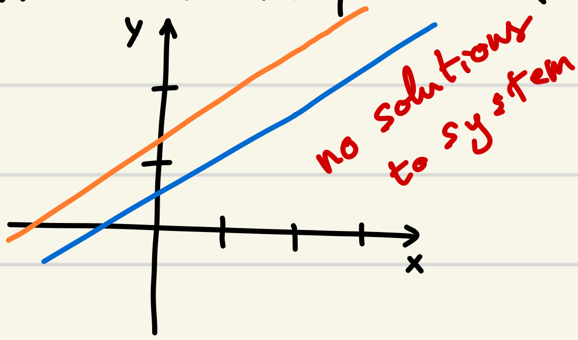
(1) the lines are not parallel: there is 1 solution



(2) the lines are parallel & equal: ∞ number of solutions



(3) the lines are parallel & not equal: no solutions



DEFINITION If a system has at least 1 solution we call it consistent and otherwise we call it inconsistent.

How to solve a system of linear equations

→ **METHOD OF ELIMINATION**

The method of elimination is based on the fact that the following operations don't change the solutions to a system of equations:

(1) Swapping 2 equations. This only changes the position of the eqns in the list but not their solutions

EXAMPLE $\begin{cases} 5x+7y=3 \\ -x+y=4 \end{cases}$ has same solutions as $\begin{cases} -x+y=4 \\ 5x+7y=3 \end{cases}$

(2) Multiplying an equation by a NONZERO constant. Since, later on, you can always divide by that constant again to retreive the old equation, this doesn't change its solutions.

EXAMPLE $\begin{cases} 5x+7y=3 \\ -x+y=4 \end{cases}$ has same solutions as $\begin{cases} 10x+14y=6 \\ -x+y=4 \end{cases}$

(3) Adding a multiple of one equation to another while keeping the original equation intact. This is essentially the same as saying: you can add the same number to both sides of an equation without changing its solutions.

EXAMPLE $\begin{cases} 5x+7y=3 \\ -x+y=4 \end{cases}$ has same solutions as $\begin{cases} 5x+7y=3 \\ 4x+8y=7 \end{cases}$ ← Eqn1 + Eqn2

Even though we're not adding numbers, but rather expressions involving variables that look different on both sides of the equation, (like $5x+7y$ and 3) we are given that these are equal ($5x+7y=3$) and its therefore perfectly fine to add these "different looking" terms to both sides of the equation.

Using these operations we can eliminate variables from equations in a systematic way

EXAMPLES

(1) Solve $\begin{cases} x-2y=8 & (E1) \\ 5x+3y=1 & (E2) \end{cases}$

By subtracting 5 times the first equation from the second we get

$$\begin{cases} x - 2y = 8 \\ 5x + 3y = 1 \end{cases} \xrightarrow{(E2) - 5(E1)} \begin{cases} x - 2y = 8 & (E1) \\ 0\cancel{x} + 13y = -39 & (E2) \end{cases}$$

Now we can multiply both sides of (E2) by $\frac{1}{13}$ to get

$$\begin{cases} x - 2y = 8 \\ 13y = -39 \end{cases} \xrightarrow{\frac{1}{13}(E2)} \begin{cases} x - 2y = 8 \\ y = -3 \end{cases}$$

Now we can add (E2) twice to (E1) to get

$$\begin{cases} x - 2y = 8 \\ y = -3 \end{cases} \xrightarrow{(E1) + 2(E2)} \begin{cases} x = 2 \\ y = -3 \end{cases}$$

so our solution is $(x, y) = (2, -3)$

(2) Solve $\begin{cases} 2x + 6y = 4 \\ 3x + 9y = 11 \end{cases} \xrightarrow{(E1)\cdot\frac{1}{2}} \begin{cases} x + 3y = 2 \\ 3x + 9y = 11 \end{cases}$

$$\xrightarrow{(E2) - 3(E1)} \begin{cases} x + 3y = 2 \\ 0 + 0 = 5 \end{cases} ↯$$

(3) Solve $\begin{cases} 2x + 6y = 4 \\ 3x + 9y = 6 \end{cases} \xrightarrow{(E1)\frac{1}{2}} \begin{cases} x + 3y = 2 \\ 3x + 9y = 6 \end{cases}$

$$\xrightarrow{(E2) - 3(E1)} \begin{cases} x + 3y = 2 \\ 0 + 0 = 0 \end{cases}$$ ← always satisfied $\longrightarrow \{ x + 3y = 2.$ ← for each x there is 1 y that solves it

In the case where we have an ∞ number of solutions it is customary to describe the solutions using one (or more) paramer(s).

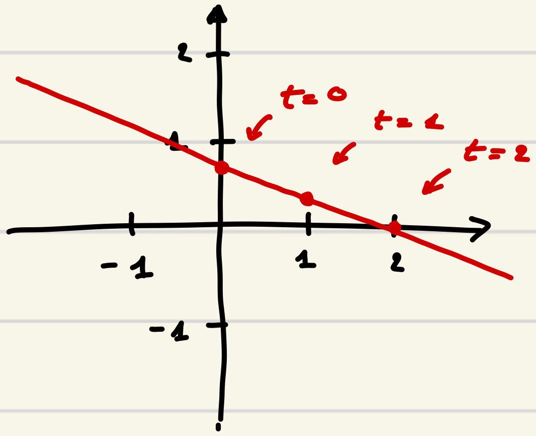
Let $x = t$ ← can be any real number, then $t + 3y = 2$ so $y = \frac{2-t}{3} = \frac{2}{3} - \frac{1}{3}t$

All solutions to the system can therefore be written as $(x,y) = (t, \frac{2}{3} - \frac{1}{3}t)$ or equivalently

couples are added piecewise

$$(x,y) = (0, \tfrac{2}{3}) + \underbrace{t\cdot(1, -\tfrac{1}{3})}_{=(t,\, -\frac{1}{3}t)}$$

$(a,b)+(c,d) = (a+c, b+d)$



Note you could also have chosen $y = t$, in which case $x = 2 - 3t$ and our solutions look like $(x,y) = (2-3t, t)$. While these look different from the solution set as a whole is the same

(4) Solve:

$$\begin{cases} x + 2y + z = 4 & E_1 \\ 3x + 8y + 7z = 20 & E_2 \\ 2x + 7y + 9z = 23 & E_3 \end{cases}$$

First we use E_1 to eliminate x from E_2 and E_3

$$\xrightarrow{\substack{(E_2) - 3(E_1) \\ (E_3) - 2(E_1)}} \begin{cases} x + 2y + z = 4 \\ 2y + 4z = 8 \\ 3y + 7z = 15 \end{cases} \xrightarrow{(E_2)\frac{1}{2}} \begin{cases} x + 2y + z = 4 \\ y + 2z = 4 \\ 3y + 7z = 15 \end{cases}$$

Now we use (E_2) to get rid of y in (E_3). We can't use (E_1) for this since then we would introduce x again!

$$\xrightarrow{(E_3) - 3(E_2)} \begin{cases} x + 2y + z = 4 \\ y + 2z = 4 \\ z = 3 \end{cases}$$

Now we can use (E_3) to eliminate z from (E_1) and (E_2)

$$\xrightarrow[(E_2) - 2(E_3)]{(E_1) - (E_3)} \begin{cases} x + 2y = 1 \\ y = -2 \\ z = 3 \end{cases} \xrightarrow{E_1 - 2(E_2)} \begin{cases} x = 5 \\ y = -2 \\ z = 3 \end{cases}$$

(5) Solve $\begin{cases} 3x - 8y + 10z = 23 \\ x - 3y + 2z = 5 \\ 2x - 9y - 8z = -11 \end{cases}$