SEC 4.3 LINEAR COMBINATIONS & INDEPENDENCE OF VECTORS.

Revision

DEF A real vector space V, is a set whose elements are called vectors which can be added together and multiplied by real numbers.

EXAMPLES (1) The set of real $n \times 1$ $\underbrace{\text{matrices}}_{\text{column vectors}}$:

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \;\middle|\; x_1, x_2, \dots, x_n \in \mathbb{R} \right\}$$

(2) The set of real $1 \times n$ matrices: row vectors

$$\left\{ \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \;\middle|\; x_1, \dots, x_n \in \mathbb{R} \right\}$$

(3) The set of real $m \times n$ matrices

$$\left\{ \begin{bmatrix} \cdot & \cdot & \cdot & \\ \vdots & & & \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \leftarrow \text{all els are real} \right\}$$

(4) The set of real differentiable functions:

$$\mathcal{C}^1(\mathbb{R}) = \{ f \mid f'(x) \text{ exists for all } x \in \mathbb{R} \}$$

(5) The set of all functions that are equal to their Taylor series:

$$\mathcal{A}(\mathbb{R}) = \left\{ f : \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in \mathbb{R} \text{ for all } i \right\}$$

DEFINITIONS

- A vector $\underline{w}$ in V is a linear combination of the vectors $\underline{v_1}, \dots, \underline{v_m}$ if and only if there exist numbers $c_1, \dots, c_m \in \mathbb{R}$ such that

$$c_1 \underline{v_1} + c_2 \underline{v_2} + \dots + c_m \underline{v_m} = \underline{w}$$

- The span of a set of vectors $\underline{v_1}, \dots, \underline{v_m}$ equals

$$\operatorname{span}\{\underline{v_1}, \dots, \underline{v_m}\} = \langle \underline{v_1}, \dots, \underline{v_m} \rangle = \{ c_1 \underline{v_1} + \dots + c_m \underline{v_m} \}$$

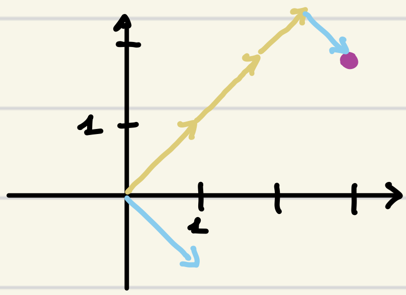
where the $c_1, \dots, c_m$ could be any real number.

EXAMPLES (1) The vector $\boxed{\underline{w} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}}$ is a linear combination of the vectors $\boxed{\underline{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ and $\boxed{\underline{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$. Indeed,

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{5}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\left(= \begin{bmatrix} \frac{5}{2} \\ \frac{5}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{6}{2} \\ \frac{4}{2} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right)$$

Geometrically



(2) $\operatorname{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\} = \mathbb{R}^2$

(3) $\operatorname{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\right\} = \mathbb{R}^2$

- How to check whether $\underline{w}$ is a linear combination of $\underline{v}_1, \dots, \underline{v}_m$?
- How to find all $c_1, \dots, c_m$ s.t. $\underbrace{c_1 \underline{v}_1 + \dots + c_m \underline{v}_m = \underline{w}}$

For $\underline{w}, \underline{v}_1, \dots, \underline{v}_m$ belong to $\mathbb{R}^n$ then comes down to

$$c_1 \begin{bmatrix} \underline{v}_1 \end{bmatrix} + c_2 \begin{bmatrix} \underline{v}_2 \end{bmatrix} + \dots + c_m \begin{bmatrix} \underline{v}_m \end{bmatrix} = \begin{bmatrix} \underline{w} \end{bmatrix}$$

or equivalently

$$\underbrace{\left[\begin{array}{c:c:c:c} \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_m \end{array}\right]}_{\underline{\underline{A}}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}}_{\underline{c}} = \begin{bmatrix} \underline{w} \end{bmatrix}$$

then this is equivalent to solving $\underline{\underline{A}}\,\underline{c} = \underline{w}$ for $\underline{c}$

EXAMPLE

To find c_1, c_2 st. $\begin{bmatrix}3\\2\end{bmatrix} = c_1 \begin{bmatrix}1\\1\end{bmatrix} + c_2 \begin{bmatrix}1\\-1\end{bmatrix}$

we solve

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\underline{\underline{A}}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Since $\underline{\underline{A}}^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$,

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

DEFINITION A set of vectors $\{\underline{v}_1, \dots, \underline{v}_m\}$ is called linearly independent (LI) if the homogenous system $c_1 \underline{v}_1 + \dots + c_m \underline{v}_m = \underline{0}$ $\circledast$ only has the trivial solution: $c_1 = c_2 = \dots = c_m = 0$

Note If e.g. $c_1 \neq 0$ in such a combination, then we can divide $\circledast$ by c_1 and rewrite it as

$$\underline{v}_1 = \left(-\frac{c_2}{c_1}\right)\underline{v}_2 + \left(-\frac{c_3}{c_1}\right)\underline{v}_3 + \dots + \left(-\frac{c_m}{c_1}\right)\underline{v}_m$$

A set of vectors $\{\underline{v}_1, \dots, v_m\}$ is LI if the homogenous system

$$\begin{bmatrix} \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_m \end{bmatrix}\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

only has 1 solution.

THEOREMS

- n vectors $\underline{v_1}, \dots, \underline{v_n}$ in $\mathbb{R}^n$ are LI if and only if

$$\det \begin{bmatrix} \vdots & \vdots & & \vdots \\ \underline{v_1} & \underline{v_2} & \cdots & \underline{v_n} \\ \vdots & \vdots & & \vdots \end{bmatrix} \neq 0$$

- Let $k > n$ then any set of k vectors $\underline{v_1}, \dots, \underline{v_k}$ in $\mathbb{R}^n$ are linearly dependent
- Let $k < n$ then the k vectors $\underline{v_1}, \dots, \underline{v_k}$ are LI if and only if

$$\begin{bmatrix} \vdots & & \vdots \\ \underline{v_1} & \cdots & \underline{v_k} \\ \vdots & & \vdots \end{bmatrix}$$

contains a $k \times k$ submatrix (obtained from leaving $n-k$ rows out) whose determinant is non zero.

SEC 4.4. BASES & DIMENSION FOR VECTOR SPACES

DEF A set of vectors $\{\underline{v_1}, \dots, \underline{v_m}\}$ form a basis of a vector space V if ANY vector in V is a UNIQUE linear combination of the set $\{\underline{v_1}, \dots, \underline{v_m}\}$

Note Another, equivalent, definition is that a basis is a subset of V that spans V and is LI.

EXAMPLES (1) For $\mathbb{R}^n$ the following is called the standard basis

$$\underline{e_1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \underline{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \underline{e_n} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

(2) A (non standard) basis for $\mathbb{R}^2$ is, e.g. $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

(3) The professor knows no example of bases for $\mathcal{C}^1(\mathbb{R})$ or $\mathcal{A}(\mathbb{R})$

Some standard questions that you will often encounter:

(1) Given a set of vectors $\underline{v_1}, \ldots, \underline{v_m}$ in $\mathbb{R}^n$. Find a basis that spans these vectors.

(2) Given a set of vectors $\underline{v_1}, \ldots, \underline{v_m}$, construct a basis for $\text{span}\{\underline{v_1}, \ldots, \underline{v_m}\}$ using only the vectors from $\{\underline{v_1}, \ldots, \underline{v_m}\}$

(3) How do you find a basis for the vector space spanned by solutions to a system of homogenous linear equations.

SOLUTIONS

(1) For example: find a basis for the space spanned by

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 3 \\ 2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 7 \\ 4 \\ 4 \end{bmatrix}$$

To find a basis write all of these vectors as row vectors in a matrix $\underline{\underline{A}}$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 2 \\ 4 & 7 & 4 & 4 \end{bmatrix}$$

and bring it in RREF

$$\xrightarrow{\substack{R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 - 4R_1}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{bmatrix} \xrightarrow{\substack{R_3 - R_2 \\ R_4 - 3R_2}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now you can read off the basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$