SEC 3.6. DETERMINANTS

There is a nice and quick formula for computing the inverse of a 2×2 matrix $\underline{\underline{A}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In the expression above $ad-bc$ determines whether $\underline{\underline{A}}$ is invertible or not and is therefor called the determinant of $\underline{\underline{A}}$.

Notations for determinant: $\det(\underline{\underline{A}}) = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$

The determinant can also be defined for general $n\times n$ matrices and it keeps its useful property, namely

> an $n\times n$ matrix is invertible if and only if its determinant is different from 0.

DEFINITIONS

Let $\underline{\underline{A}}$ be an $n\times n$ matrix,

- the ij^{th} minor of $\underline{\underline{A}}$, denoted by M_{ij}, is the determinant of the submatrix of $\underline{\underline{A}}$ obtained by removing the i^{th} row and j^{th} column of $\underline{\underline{A}}$.

EXAMPLE M_{22} for $\underline{\underline{A}} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & \ell \end{bmatrix}$

is equal to $\det \begin{bmatrix} a & \cancel{b} & c \\ \cancel{d} & \cancel{e} & \cancel{f} \\ g & \cancel{h} & \ell \end{bmatrix} = \det \begin{bmatrix} a & c \\ g & \ell \end{bmatrix} = a\ell - gc$

- The ijth cofactor of $\underline{\underline{A}}$, written A_{ij}, and it equals $A_{ij} = (-1)^{i+j} M_{ij}$

  To visualize $(-1)^{i+j}$, you can use a checkerboard pattern as follows

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & & & & \end{bmatrix}$$

- The determinant of an $n \times n$ matrix can be computed by choosing a row, say the k'th row, or a column, say the ℓth column as follows:
  - ▶ If you use the kth row then
  $$\det(\underline{\underline{A}}) = [\underline{\underline{A}}]_{k1}(-1)^{k+1} M_{k1} + [\underline{\underline{A}}]_{k2}(-1)^{k+2} M_{k2} + \cdots + [\underline{\underline{A}}]_{kn}(-1)^{k+n} M_{kn}$$
  - ▶ If you chose the ℓth column
  $$\det(\underline{\underline{A}}) = [\underline{\underline{A}}]_{1\ell}(-1)^{1+\ell} M_{1\ell} + [\underline{\underline{A}}]_{2\ell}(-1)^{2+\ell} M_{2\ell} + \cdots + [\underline{\underline{A}}]_{n\ell}(-1)^{n+\ell} M_{n\ell}$$

EXAMPLES

(1) Compute $\det(\underline{\underline{A}})$ by the cofactor expansion of the first row, where

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\det(\underline{\underline{A}}) = [\underline{\underline{A}}]_{11} M_{11} - [\underline{\underline{A}}]_{12} M_{12} + [\underline{\underline{A}}]_{13} M_{13}$$

$$= 1 \cdot M_{11} - 2 M_{12} + 3 M_{13}$$

$$M_{11} = \det \begin{bmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ \cancel{4} & 5 & 6 \\ \cancel{7} & 8 & 9 \end{bmatrix} = \det \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} = 9 \cdot 5 - 8 \cdot 6 = -3$$

$$M_{12} = \det \begin{bmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ 4 & \cancel{5} & 6 \\ 7 & \cancel{8} & 9 \end{bmatrix} = \det \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} = -6$$

$$M_{13} = \det \begin{bmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ 4 & 5 & \cancel{6} \\ 7 & 8 & \cancel{9} \end{bmatrix} = \det \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix} = 4 \cdot 8 - 7 \cdot 5 = -3$$

$$\det(\underline{\underline{A}}) = 1 \cdot (-3) - 2(-6) + 3(-3) = 0$$

(2) Compute $\det(\underline{\underline{A}})$ for

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 4 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

We'll expand using the second column:

$$\det(\underline{\underline{A}}) = 0 M_{12} + 1 \cdot M_{22} + 0 M_{32} + 0 \cdot M_{42}$$

$$= M_{22}$$

$$= \det \begin{bmatrix} 1 & \cancel{0} & 0 & 3 \\ \cancel{0} & \cancel{1} & \cancel{1} & \cancel{0} \\ 2 & \cancel{0} & 4 & 1 \\ 1 & \cancel{0} & 1 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 3 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\overset{\substack{\text{first row} \\ \text{expansion}}}{=} 1 \cdot M_{11} - \cancel{0 \cdot M_{12}} + 3 M_{13}$$

$$= 1 \cdot \det \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} + 3 \det \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} = 4 \cdot 1 - 1 \cdot 1 + 3(2 \cdot 1 - 1 \cdot 4)$$

$= 3 + 3(-2) = \boxed{-3}$

PROPERTIES

(1) If a matrix $\underline{\underline{B}}$ is obtained from $\underline{\underline{A}}$ by multiplying one of its rows or columns by a constant c, then $\det(\underline{\underline{B}}) = c\det(\underline{\underline{A}})$

EXAMPLE

$$\det\begin{vmatrix} a & b & 3c \\ d & e & 3f \\ g & h & 3i \end{vmatrix} = 3\det\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

(2) If $\underline{\underline{B}}$ is the matrix obtained from $\underline{\underline{A}}$ by swapping a row or a column, then

$$\det(\underline{\underline{B}}) = -1 \cdot \det(\underline{\underline{A}})$$

(3) If 2 rows or columns of $\underline{\underline{A}}$ are a multiple of each other, $\det(\underline{\underline{A}}) = 0$.

EXAMPLE

$$\det\begin{vmatrix} a & 2a & b \\ c & 2c & d \\ e & 2e & f \end{vmatrix} = 0$$

(4) $$\det\begin{vmatrix} a_1+a_2 & b & c \\ d_1+d_2 & e & f \\ g_1+g_2 & h & i \end{vmatrix} = \det\begin{vmatrix} a_1 & b & c \\ d_1 & e & f \\ g_1 & h & i \end{vmatrix} + \det\begin{vmatrix} a_2 & b & c \\ e_2 & e & f \\ g_2 & h & i \end{vmatrix}$$

this result holds for any row or column for matrices of any size.

(5) Let $\underline{\underline{B}}$ be a matrix obtained from $\underline{\underline{A}}$ by adding a multiple of one row of $\underline{\underline{A}}$ to another one. Then $\det(\underline{\underline{B}}) = \det(\underline{\underline{A}})$.
The same holds for columns.

$$\det \begin{bmatrix} a & b & c \\ d+2a & e+2b & f+2c \\ g & h & i \end{bmatrix} = \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \cancel{\det \begin{bmatrix} a & b & c \\ 2a & 2b & 2c \\ g & h & i \end{bmatrix}}$$

(6) DEFINITIONS A matrix is called upper triangular if all elements below the main diagonal are 0.

EXAMPLES $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \end{bmatrix}$

A matrix is lower triangular if all elements above the main diagonal are 0.
A matrix is triangular if its either upper or lower triangular.
Note a matrix in REF is upper triangular.

The determinant of a triangular matrix equals the product of the elements on the diagonal.

EXAMPLE

$$\det \begin{bmatrix} a & b & c & d & e \\ 0 & f & g & h & i \\ 0 & 0 & j & k & l \\ 0 & 0 & 0 & m & n \\ 0 & 0 & 0 & 0 & p \end{bmatrix} = a f j m p$$

(7) **DEFINITION** The transpose $\underline{\underline{A}}^T$ of $\underline{\underline{A}}$ is the matrix with elements

$$[\underline{\underline{A}}^T]_{ij} = [\underline{\underline{A}}]_{ji}$$

EXAMPLE

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}^T = \begin{bmatrix} a & e \\ b & f \\ c & g \\ d & h \end{bmatrix}$$

$$\det(\underline{\underline{A}}^T) = \det(\underline{\underline{A}})$$

8) $\det(\underline{\underline{A}}\,\underline{\underline{B}}) = \det(\underline{\underline{A}})\det(\underline{\underline{B}})$ if $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are $n \times n$ matrices.

THEOREM CRAMER'S RULE

Given a system of lin eqns

$$\underline{\underline{A}} \cdot \underline{x} = \underline{b} \quad \text{with} \quad \underline{\underline{A}} = \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \cdots & \underline{a}_n \end{bmatrix}$$

column vectors

then

$$x_i = [\underline{x}]_i = \frac{\det\begin{bmatrix} \underline{a}_1 & \cdots & \underline{a}_{i-1} & \underline{b} & \underline{a}_{i+1} & \cdots & \underline{a}_n \end{bmatrix}}{\det(\underline{\underline{A}})}$$