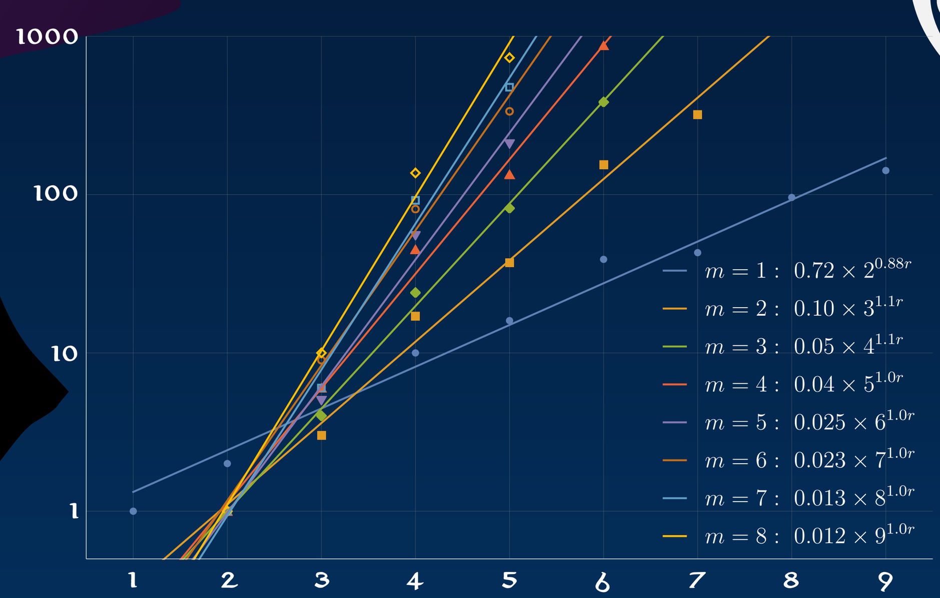
Finding fusion rings

Finding fusion rings comes down to finding 3D tables N[a,b,c] of structure constants that satisfy the defining properties. For every table N[a,b,c] there are N! tables, corresponding to permutations of the particles, that give rise to an equivalent ring. The main ingredients to the search algorithm we used are
(i) reduction of variables,
(ii) permutation symmetry breaking,
(iii) simplifying and sorting the associativity equations, and
(iv) using a tree search to find solutions to these equations

Results

A total of 28541 fusion rings have been found. 118 of these are non commutative, 54 are modular. Interestingly, when interpolating the number of fusion rings per rank and multiplicity the rates of growth are given by $C(m+1)^{br}$ where $b \approx 1$.



Fusion rings

appear in various situations in mathematics and physics where a combination of two objects returns an integer amount of those objects. A classical example is the decomposition of a tensor product of irreps into a sum of irreps. Another prominent example is the study of anyon models, which have profound applications in the field of topological quantum computation and topological phases of matter.

A Fusion Ring Consists of :

a finite set of elements (also called particles, or charges)

$$\{\psi_1, \ldots, \psi_n\}$$

with an associative bilinear product

$$\psi_a \times \psi_b = \sum_c N_{ab}^c \psi_c, \quad N_{ab}^c \in \{0, 1, \ldots, m\}$$

✦ ✦ ✦ such that ✦ ✦ ✦

there is a unit 1 (also called the vacuum)

$$1 \times \psi_a = \psi_a = \psi_a \times 1$$

every particle ψ_a has a unique conjugate (anti-)particle $\psi_{\bar{a}}$

$$N_{a\bar{a}}^1 = 1 = N_{\bar{a}a}^1$$

the structure constants obey pivotal relations

$$N_{ab}^c = N_{\bar{a}c}^b = N_{b\bar{c}}^{\bar{a}}$$

Songs

SONG (Single Orbit Normal subGroup) extensions are fusion rings with a subgroup whose action on the rest of the ring is nice. Songs are a generalization of the Tambara-Yamagami and Haagerup-Izumi constructions. The former don't have an associated fusion categories for non-commutative groups, while the latter are only defined for commutative groups. We showed there are songs based on the group D_3 that do have categorifications, though. The multiplication table of one of those is displayed.

1	2	3	4	5	6	7	8
2	1	6	5	4	3	8	7
3	5	1	6	2	4	8	7
4	6	5	1	3	2	8	7
5	3	4	2	6	1	7	8
6	4	2	3	1	5	7	8
7	8	8	8	7	7	$H_7^l + \sigma$	$2 \times H_7^l + \sigma$
8	7	7	7	8	8	$2 \times H_7^l + \sigma$	$H_7^l + \sigma$

$H_7^l = 1 + 5 + 6 \quad \sigma = 7 + 8$

Topological Quantum Computing

Fusion rings describe splitting and fusion of anyons: particles living in 2D whose exchange is governed by representations of the braid group. Probabilities of processes where anyons braid are obtained using link invariants: classically an exponentially hard computation, but now just a matter of moving particles. Kitaev et al. also showed any classical computation can be performed by braiding anyons. The computations are moreover topologically protected since local perturbations in knots don't affect invariants.

Where to go from here?

To aid researchers studying anyon models, fusion rings, and fusion categories the AnyonWiki was created: an open database of fusion rings, fusion categories, and pages on related concepts. Furthermore a Wolfram Language package, FusionRings, was created to work with fusion rings and probe their properties. The next big step is to categorify as many fusion rings as possible and create tools for dealing with these fusion categories. The AnyonWiki should have all data on these fusion categories in a computational format. Currently a package, FusionCategories, is in the making and the first categories are rolling out of the computer.

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On low rank Fusion Rings