ON LOW-RANK MULTIPLICITY-FREE

FUSION CATEGORIES

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MISSION OBJECTIVES

- Complete the classification of all (braided) (pivotal) multiplicity-free fusion categories up to rank 7.
- Create a package containing
  - all explicit data of these categories,
  - functions for working with fusion categories, and
  - functions for finding more fusion categories.
- Make the data availlable online for other researchers.

ARSENAL - ANYONICA

Anyonica is a Mathematica package that provides an arsenal of tools to deal with pentagram and hexagram equations.

- FindZeroValues
  Finds all possible configurations of zero values of F-symbols.
- BreakMultiplicativeSymmetry
  Breaks all continuous gauge symmetry by fixing values of F-symbols.
- ReduceByBinomials
  Recursively updates a system of polynomial equations by solving equations with two terms.
- ReduceByLinearity
  Recursively updates a system of polynomial equations by solving linear equations.
- IncrementalGroebnerBases
  Calculates a Groebner basis in an incremental manner.
- SolveViaReduce
  Solves a system of polynomial equations using Mathematica's Reduce.

$a \otimes (b \otimes (c \otimes d))$ $((a \otimes b) \otimes c) \otimes d$

FUSION CATEGORIES

provide a powerful language transcending traditional disciplinary boundaries. Their applications range from the abstract domains of algebra, topology, category theory, and representation theory to the practical disciplines of quantum information and topological quantum computation. Despite their usefulness, there is a significant lack of examples of such categories and an even greater lack of explicit computational data with which to experiment. The goal of our research is to find and share as many examples of multiplicity-free fusion categories as possible.

PIVOTAL & SPHERICAL FUSION CATEGORIES

come with a pivotal structure

$$\psi : x \xrightarrow{\sim} x^{**},$$

a natural tensor isomorphism that maps objects and morphisms to their double dual. To find all pivotal structures one must solve the

PIVOTAL EQUATIONS

$$\frac{p_c}{p_a p_b} = [F_1^{abc^*}]_{a^*}^{c} [F_1^{bc^*a}]_{b^*}^{a^*} [F_1^{c^*ab}]_{c}^{b^*}$$

These are useful to calculate the quantum dimensions

$$d_a = p_a / [F_a^{aa^*a}]_1^1$$

a strong set of invariants of the fusion category. If $d_a = d_{a^*}$ for all objects a, then the category is called a spherical fusion category. These are useful to calculate 3-manifold invariants. If also braided, one can also extract knot invariants.

A fusion category

over $\mathbb{C}$ is a category that is $\mathbb{C}$−linear, Abelian, indecomposable, semi−simple, finite, monoidal with product $\otimes$ that is bilinear on morphisms, rigid, and for which $\mathrm{End}(1) \cong \mathbb{C}$. To find fusion categories one must solve the

PENTAGRAM EQUATIONS

$$[F_e^{fcd}]_l^g [F_e^{abl}]_k^f = \sum_h [F_g^{abc}]_h^f [F_e^{ahd}]_k^g [F_k^{bcd}]_l^h$$

a third-degree polynomial system of monstrous size. No standard software could deal with systems of this size so we wrote a package, Anyonica, to solve these equations. Anyonica also contains a list of all multiplicity-free fusion categories up to rank 7 and other useful functions for working with fusion rings and categories.

BRAIDED FUSION CATEGORIES

come with a braided structure:

$$\beta : a \otimes b \to b \otimes a$$

This structure provides representations of the braid group and thus also solutions to the Yang-Baxter equations. To find all braided structures one must solve the

HEXAGRAM EQUATIONS

$$R_e^{ca} [F_d^{acb}]_g^e R_g^{cb} = \sum_f [F_d^{cab}]_f^e R_d^{cf} [F_d^{abc}]_g^f$$

$$\tilde{R}_e^{ca} [F_d^{acb}]_g^e \tilde{R}_g^{cb} = \sum_f [F_d^{cab}]_f^e \tilde{R}_d^{cf} [F_d^{abc}]_g^f$$

Despite the system being smaller in size, the coefficients of these equations, the F-symbols, are solutions to the Pentagram equations and can be quite troublesome.

UNITARY FUSION CATEGORIES

come with a Hermitian structure, $\dagger$, that induces a canonical spherical structure for which the quantum dimensions are positive real numbers. For a unitary fusion category there always exists a basis for which the F-matrices are unitary. Having unitary F-matrices only implies that the category is unitary if equiped with the canonical spherical structure. Unitarity is often required for applications to anyonic systems in low-dimensional quantum physics. Here, the evolution of such a system is described by the morphisms of a unitary (modular) fusion category and unitarity guarantees conservation of probability. For mathematical purposes, such as the calculation of topological invariants, unitarity is not required, so we never assumed such property when searching for fusion categories.

MODULAR FUSION CATEGORIES

are spherical braided fusion categories for which the S-matrix

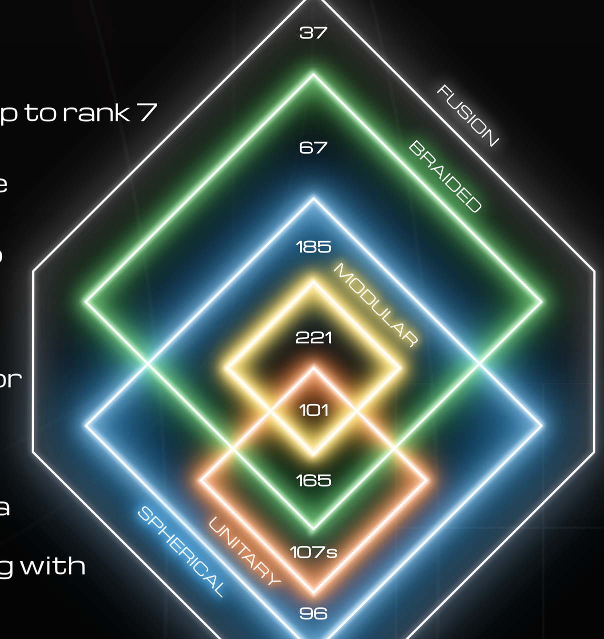
$$[S]_b^a = d_a d_b \sum_c [\tilde{F}_a^{ab^*b}]_c^1 R_c^{b^*a} R_c^{ab^*} [F_a^{ab^*b}]_1^c$$

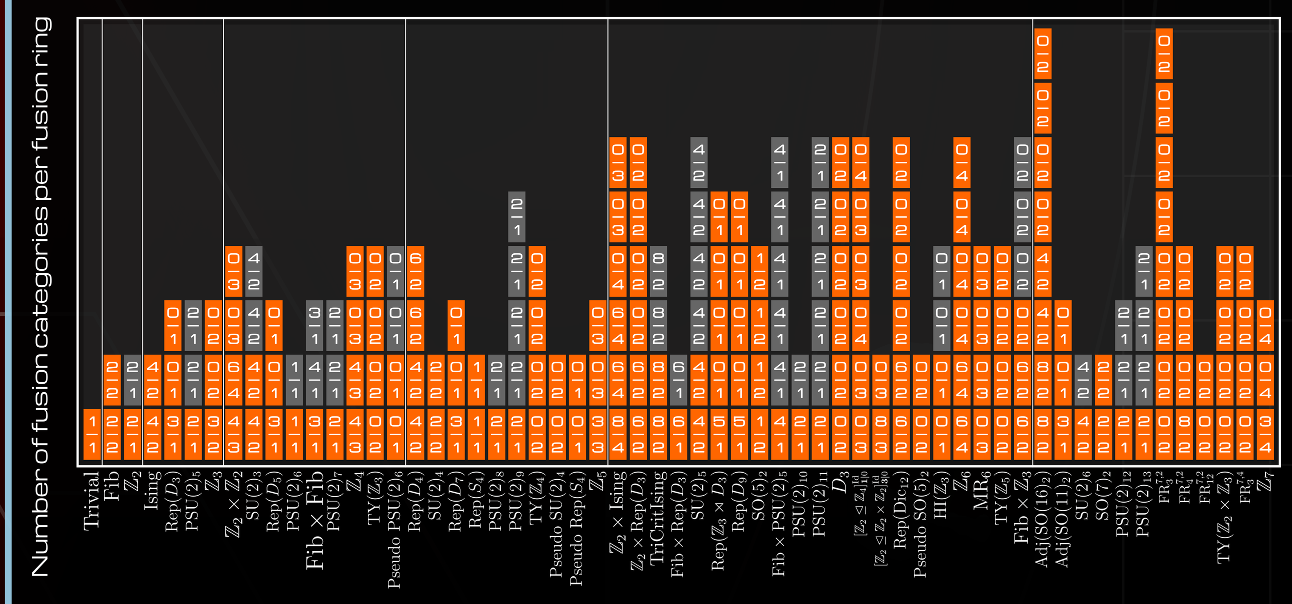
is invertible. The name comes from the fact that such a fusion category provides a representation of the modular group $\mathrm{SL}(2,\mathbb{Z})$. Unitary modular fusion categories model anyonic quantum systems that provide means to perform topological quantum computation. This is a form of quantum computation where calculations are performed braiding anyons around each other. The outcome of such computations is independent of the trajectory of the anyons and is therefore protected against perturbations. The chances that specific processes occur in the lab are related to knot invariants, which can be calculated exponentially faster this way.

$(p \otimes (q \otimes r))$ $(p \otimes q) \otimes r$ $(p \otimes q) \otimes (q \otimes r)$

RESULTS

- All multiplicity-free fusion categories (973 in total) up to rank 7 have been found. The number of categories with specific properties are shown in the Venn diagram. The number of categories per fusion ring are shown in the bar chart.
- GENERAL PROPERTIES
  - Every multiplicity-free fusion category up to rank 7 admits a pivotal structure.
  - Not all fusion rings with modular data are categorifiable.
  - If a multiplicity-free modular with rank up to 7 is categorifiable, then it has at least one modular category.
  - Some fusion categories of rank 7 have different configurations of zero values for the F-symbols. All these configurations form a chain of inclusions, however.
- All the data on fusion rings and categories are part of the Anyonica package. Anyonica also contains all methods for finding fusion categories and useful functions for working with these rings and categories.
- All F-symbols and R-symbols are available online on the AnyonWiki.





MISSION OVERVIEW

"Cutting off the sources of the invasion as soon as possible was as hard as it was useful. In the end, only in a few cases did we manage to avoid a full invasion at this point."

1 To break symmetry and reduce the system we need to know which variables can be zero. This is done by mapping the system to a logical proposition: we map each variable to a boolean which is False if the variable is zero and True otherwise. Each polynomial equation becomes a proposition saying it is impossible for exactly one term to be non-zero. The invertibility of the pentagram maps becomes the proposition that the determinant polynomials of the F-matrices must have at least one non-zero term.
Finding all zero values then reduces to an ALLSAT problem from computer science: find all boolean vectors satisfying a proposition.

"What used to be pure energy flowing down the tower now had a somewhat more `macabre' nature."

2 Once the zeros are known we start breaking symmetry. For every solution $\{[F_d^{abc}]_f^e\}$ to the pentagram equations there are an infinite number of other solutions of the form

$$\left\{ \frac{g_e^{ab} g_d^{ec}}{g_d^{af} g_f^{bc}} [F_d^{abc}]_f^e \right\}$$

Since none of the F-symbols is zero, one can set F-symbols with freedom to 1 and solve for the gauge variables.

"It didn't help that the accelerator converted souls to what looked like an infinity of them"

3 The remaining system is often too big to tackle so we use subsystems of a simpler form to to simplify the harder equations. First the equations with only two terms are used. Then the equations that are linear in a variable, are used. Often assumptions need to be added and the number of systems to solve grows.

"Every level of the facility had to be cleared one by one. We'd like to say we never saw the poor souls back that faced the whole lot in one go..."

Danger Level: CRITICAL!

Chance of Overflow: HIGH!

4 Once all the simplest equations are exhausted, the final step consists of finding a Groebner basis. Although the system is much smaller in terms of Jnumber of equations and variables, some equations have evolved to monstrous size. By calculating the Groebner bases of subsystems these monstrous equations often break down at some point.

"We can only pray we'll make it to"

5 To complete the mission, we must find the roots of the Groebner basis and use back-substitution to obtain the values of all F-symbols.
This procedure must also be applied to all Hexagram equations. Fortunately, the Pivotal equations pose no threat at all.

"There's a very thin line between ambition and stupidity, especially when greed comes into play. What should have been a single trip to Hell became a routine to be repeated more than a thousand times."

Anyonica

AnyonWiki

Poster content is based on the Ph.D. of Gert Vercleyen under supervision of Joost Slingerland.

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